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Universal topological phase of two-dimensional stabilizer codes

H Bombin¹, Guillaume Duclos-Cianci² and David Poulin^{2,3}

¹ Perimeter Institute for Theoretical Physics, 31 Caroline St. N, Waterloo, ON, N2L 2Y5, Canada

² Département de Physique, Université de Sherbrooke, Sherbrooke, Québec, J1K 2R1, Canada

E-mail: David.Poulin@USherbrooke.ca

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Abstract. Topological phases can be defined in terms of local equivalence: two systems are in the same topological phase if it is possible to transform one into the other by a local reorganization of its degrees of freedom. The classification of topological phases therefore amounts to the classification of long-range entanglement. Such local transformation could result, for instance, from the adiabatic continuation of one system's Hamiltonian to the other. Here, we use this definition to study the topological phase of translationally invariant stabilizer codes in two spatial dimensions, and show that they all belong to one universal phase. We do this by constructing an explicit mapping from any such code to a number of copies of Kitaev's code. Some of our results extend to some two-dimensional (2D) subsystem codes, including topological subsystem codes. Error correction benefits from the corresponding local mappings. In particular, it enables us to use decoding algorithm developed for Kitaev's code to decode any 2D stabilizer code and subsystem code.

³ Author to whom any correspondence should be addressed.

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1. Introduction

The theory of Ginzburg and Landau has had tremendous success in classifying the different phases of matter in terms of local order parameters and spontaneously broken symmetries. However, it fails to classify certain states of nature, such as the different fractional quantum Hall fluids that all have the same local symmetries. The Hamiltonian of these systems has a constant energy gap, and the ground state degeneracy depends on the topology of space. Crucially, all ground states are locally identical, which explains the failure of the Ginzburg–Landau paradigm. Instead, the classification of these systems requires the concept of topological order.

Since topological order reflects the long-scale many-body correlations of the system, it cannot be modified locally. This robustness [1–4] is indeed one of the many features that makes topologically ordered systems interesting for quantum information processing [5]. It also suggests a natural classification of topological phases: systems that only differ by a local rearrangement of their degrees of freedom belong to the same topological phase. In other words, the different phases are characterized only by their long-range entanglement patterns [6].

Another, more conventional, description of these phases is in terms of adiabatic connections. If two local and gapped Hamiltonians are connected by a family of local and gapped Hamiltonians, then it should be possible to adiabatically interpolate between the two without encountering a phase transition. The two systems should therefore be in the same phase. This adiabatic evolution will generate a local unitary transformation [7]; so consequently the two systems will be in the same topological phase according to the definition adopted above.

Quantum error-correcting codes [8] are intimately related to topological order. To protect the information from local errors, information is encoded into the long-range entanglement of the system. A stabilizer code [9] is a special type of quantum code that can be defined as the degenerate ground state of a Hamiltonian on N qubits of the form

$$H = - \sum_a S_a, \quad \text{with } [S_a, S_b] = 0 \quad \forall a, b, \quad (1)$$

where the stabilizer operators S_a are Hermitian elements of the Pauli group, i.e. they are constructed from tensor products of the three Pauli matrices σ_x , σ_y and σ_z and the identity operator I . Stabilizer codes are also *frustration free*, meaning that the S_a do not generate

-1 under multiplication; so the ground states of H are $+1$ eigenstates of all stabilizers, i.e. $S_a|\psi\rangle = +|\psi\rangle$ for all a . The S_a form an Abelian group under multiplication, the stabilizer group \mathcal{S} . When the qubits are embedded on a regular lattice, the code—or its associated Hamiltonian—is said to be *local* if each operator S_a has support on a region of constant size, independent of the system size. The support of an operator contains those qubits on which it acts non-trivially.

In this paper, we are interested in stabilizer codes that (i) are local and translationally invariant (LTI), and (ii) are topological, in the sense that no local operator can recover any encoded information—i.e. they have a macroscopic minimum distance in terms of error correction or they have no local order parameter in terms of many-body physics. If we place our stabilizer in an infinite lattice, this can be formalized as follows.

Definition 1. A *topological stabilizer code (TSC)* is an LTI stabilizer \mathcal{S} such that $\mathcal{Z}(\mathcal{S}) \propto \mathcal{S}$.

The symbol $\mathcal{Z}(\mathcal{S})$ denotes the centralizer of \mathcal{S} , the group of Pauli operators (with bounded support) that commute with all the elements of \mathcal{S} . Our main result is that the topological phase of any 2D TSC is uniquely determined by its total quantum dimension $D = 2^n$, or equivalently by its topological entanglement entropy $S_{\text{topo}} = n \log 4$ [10, 11]. This follows from the existence of a local mapping to n copies of Kitaev's topological code (KTC) [2, 12]. We also adapt the result to a class of subsystem stabilizer codes [13, 14].

Many considerations motivate this line of research. Firstly, stabilizer codes provide simple models to study many-body quantum physics because they often admit exact solutions, and at the same time can exhibit complex phenomena such as topological order and anyonic excitations [2, 12, 15]. To our knowledge, this is the first example where the definition of topological order based on local equivalence [6] can be directly applied to a class of models in a rigorous manner. Secondly, in the context of error correction, the local equivalence to KTC enables us to directly extend a number of properties of this code to all two-dimensional (2D) TSCs. For instance, thermal instability [16–19], code tradeoffs [20], logical operator geometry [21] and scale invariance [22] all become trivial corollaries of our mapping. In addition, our mapping provides a method to decode any 2D TSC code, while only a handful of special cases previously had solutions [12, 23, 24]. Thirdly, the local mapping can be used to change encoding during a quantum computation. As the mapping is local, this change will not propagate errors and is therefore fault-tolerant. This allows us to put together the features of different codes—such as having transversal Clifford gates [15], lower weight stabilizer generators [2, 12, 25], etc—and suggests a natural generalization of the notion of transversality for topological codes to include all local gates.

The rest of this paper is arranged as follows. Section 2 introduces basic definitions. Section 3 states our main result—the local equivalence of all TSCs—and presents a detailed construction of the mapping that realizes this local equivalence. The section ends with an illustration of the mapping for topological color codes (TCCs) [15, 25]. The following section describes how the main results extends to a class of subsystem codes [13, 14], and in particular this is illustrated with the topological subsystem color codes (TSCCs) [25]. An important application of the main result is developed in section 5, where we show how any TSC (subsystem or subspace) can be efficiently decoded, a result that extends even beyond the realm of applicability of our main result. We conclude with a brief summary in section 6.

2. Definitions

The notion of locality plays a crucial role here. For an operator X acting on the qubits of a 2D lattice, let us denote by $|X|$ the range of X , defined as the size of the smallest square containing the support of X . With this definition, a Hamiltonian of the form equation (1) H is *local* if there exists a constant w such that $|S_a| \leq w$ for all a . A translationally invariant unitary transformation U is *local* if there exists a constant v such that $|U^\dagger X U| \leq |X| + v$ for all operator X . Note that this definition is equivalent [26] to the requirement that U be decomposable into a system-size-independent sequence of nearest-neighbor unitary transformations (and possibly making use of auxiliary qubits). Lastly, we will say that two local stabilizer codes defined by Hamiltonians H and H' equation (1), with stabilizer groups \mathcal{S} and \mathcal{S}' , are *locally equivalent* if there exists a local unitary U and two *trivial* LTI stabilizer groups \mathcal{T} and \mathcal{T}' such that $U(\mathcal{S} \otimes \mathcal{T})U^\dagger = \mathcal{S}' \otimes \mathcal{T}'$. A trivial stabilizer group is generated only by single-qubit operators. Physically, U takes the ground state of H onto that of H' , and adds or removes extra qubits that are completely unentangled. The existence of renormalization group transformations that disentangle some qubits from topological codes [22] shows the necessity of \mathcal{T} and \mathcal{T}' in this definition.

KTC [2, 12] is defined on a 2D square lattice, with one qubit attached to each edge. For each lattice site s , define an operator $A_s = \prod_{e \in E_s} \sigma_z^e$, where E_s denotes the set of edges incident to site s . Similarly, define for each lattice plaquette p (site of the dual lattice) an operator $B_p = \prod_{e \in E_p} \sigma_x^e$ where E_p denotes the set of edges adjacent to plaquette p . The Hamiltonian of the model is

$$H = - \sum_s A_s - \sum_p B_p. \quad (2)$$

The excitations are anyons, gapped, and topologically charged. Indeed, any set of excitations contained in any finite region of the KTC can be reduced by local operations (i.e. acting in that region) to one of four configurations: the vacuum (0) corresponding to no excitations, an electric charge (e) corresponding to a plaquette excitation B_p , a magnetic charge (m) corresponding to a site excitation A_s , and a composite excitation (f) containing both. These four sectors are the topological charges of the model. Excitations with different charges are characterized by different topological interactions or braiding statistics. According to the effect of exchanging two identical charges, electric and magnetic particles are classified as bosons, while the composite particle is a fermion. As for mutual statistics, they are all semionic because braiding any two distinct non-vacuum charges yields a -1 phase. Finally, two charges can merge to form a new charge. The corresponding fusion rules are Abelian and such that $m \times e \rightarrow f$ and $\sigma \times \sigma \rightarrow 0$, for $\sigma = m, e, f$.

For general models of the form (1) charge is defined analogously. That is, it labels equivalence classes of excited states up to local transformations, with the understanding that excited states are assumed to be common eigenstates of all the Hamiltonian terms. The notion of topological charge is of utmost relevance because local equivalence preserves the anyon model. Indeed, the anyon model can be derived from commutation properties of certain string operators, which are unaffected by unitary conjugation.

3. The main result

We assume that 2D TSCs cannot give rise to chiral anyons⁴ because the Hamiltonian terms S_a commute with each other [27]. In our framework, the presence of chiral anyons is defined by certain properties of a matrix introduced later, in definition 2. Since this definition relies on concepts developed in this paper, we save it for later. Under the assumption that there are no chiral anyons, our main result is:

Theorem 1. *Every 2D TSC is locally equivalent to a finite number of copies of KTC.*

By n copies of the code, we mean stacking n lattices on top of each other, each with the same Hamiltonian (2). This result implies that equivalence classes are labeled by the total quantum dimension of the code. Just as the singlet (ebit) can be taken as the fundamental unit of bi-partite entanglement [28], our main result suggests that KTC can serve as the fundamental unit of long-range entanglement for TSCs. The result also demonstrates how to systematically search for new translationally invariant stabilizer codes in 2D: take any given number of copies of KT and apply any local unitary in a TI way.

We note that translational invariance is crucial to arrive at this characterization, but also the fact that we are not considering any sort of boundary conditions. Indeed, these can give rise to additional structure that could require interactions between different copies of KTC along the boundaries. A clear example is given by models where the translational symmetry of the excitations is smaller than the translational symmetry of the code. This happens, for example, in Wen's version of KTC [29] and in Bravyi's 'strange code' [30], and amounts to the possibility of introducing global topological defects, analogous to the localized topological defects described in [31].

3.1. Construction of the mapping

In this section, we give a detailed construction of the mapping between any TSC and copies of KTC. This construction is the core of our main theorem. The only details that we will leave out are (i) that given a Hamiltonian of the form equation (1), the S_a can be chosen to be independent under multiplication while preserving locality and translational invariance, and (ii) any TSC contains a finite number of topological charges. These two propositions are perhaps not so surprising, but their proofs are very cumbersome and can be found in [32], so we take them here as assumptions. It is worth noting that the first assumption rules out the existence of loop-like excitations, as those appearing in the Ising model. Indeed, loops satisfy a local conservation rule, as they cannot have endpoints, and these local conditions would imply the existence of constraints for the S_a .

Note that the first proposition generally requires coarse graining the lattice. With further coarse graining, we can also make sure that all stabilizer generators S_a have support on a 2×2 square. Most steps in the construction require coarse graining the lattice. In what follows, the description of every step assumes the lattice resulting of the coarse graining from all previous steps (although this is certainly not optimal).

⁴ In Abelian models, the chiral central charge c_- is related to the topological spins θ_i ($= 1$ for bosons, -1 for fermions) through $\sum_i \theta_i = \kappa e^{\pi i c_- / 4}$ [27].

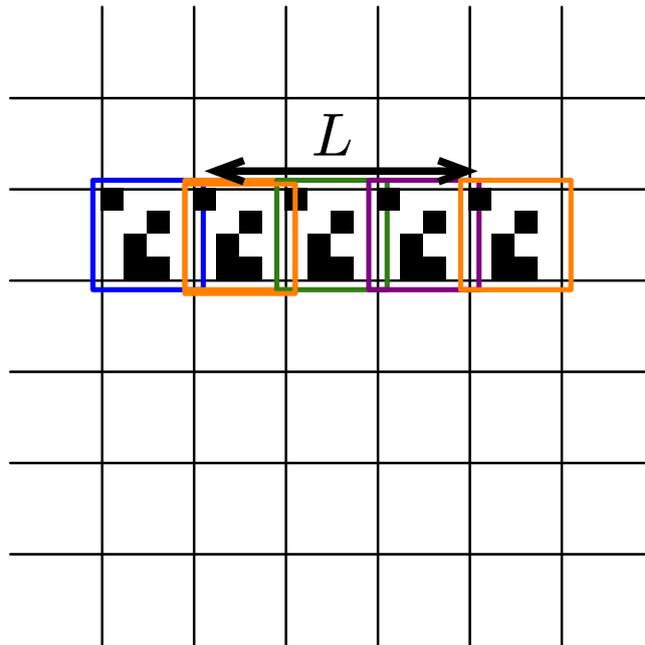


Figure 1. Black squares in a site represent an excitation configuration. The colored squares around each site represent its charge, different colors corresponding to different charges. The translations of an excitation configuration will in general modify its charge. However, because of translational invariance, it must lead to permutations of the charges. Since the number of charges is finite, one of the permutations must repeat at some point.

3.1.1. Topological charges. We begin by identifying the topological charges of the TSC at hand. Each eigenstate of the system has a set of excitations $\{S_a\}$, the Hamiltonian terms with negative eigenvalue. As in Kitaev's code, topological charges are equivalence classes of such excitation configurations under local operations. In particular, two configurations with sets of excitations $\{S_a\}$ and $\{S_b\}$ are topologically equivalent when there exists a (finite weight) Pauli operator p that anti-commutes with the Hamiltonian terms $\{S_a\} \Delta \{S_b\}$, where Δ denotes the symmetric difference of sets—we say that p has syndrome $\{S_a\} \Delta \{S_b\}$. Excitations form a group with product Δ , formalizing the notion of fusion rule here. Since in 2D there is always a finite number of topological charges and because Pauli operators square to the identity, the charge group is isomorphic to the direct product \mathbb{Z}_2^N for some integer N , e.g. $N = 2$ in KTC.

We now want to simplify the geometrical layout of the topological charges. We can attach a charge to each stabilizer generator S , namely that of the singleton $\{S\}$. Due to translational symmetry and the finiteness of the number of charges, a suitable coarse graining will make the charge TI and guarantee that every charge can be represented by a set of excitations occupying a single site. Indeed, start by coarse graining till the site at the origin contains, in this sense, all possible charges. Then the charges of the stabilizer generators at any other given site are given by a permutation of those at the origin. But the number of possible permutations is finite and thus there must be two sites α and $\alpha + L\mathbf{x} \in \mathbb{Z}^2$ separated by a horizontal distance L as in figure 1 with the same permutation. Then the charge of any stabilizer generator S_a^α and its horizontal translation $S_a^{\alpha+L\mathbf{x}}$ must be the same. The other axis is analogous, and by coarse graining we

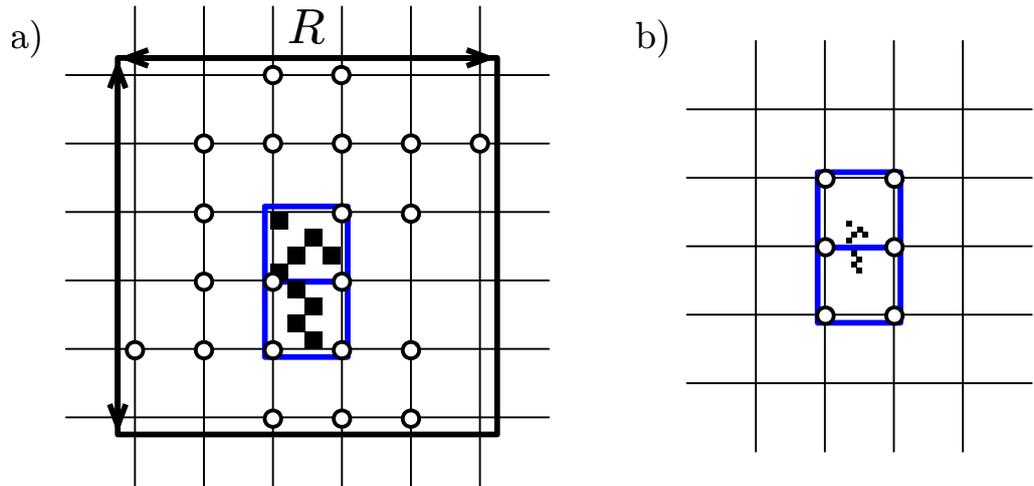


Figure 2. (a) For any two excitation configurations with the same charge on neighboring sites (or on a single site), there must exist a Pauli operator that creates these excitations, by definition of charges. The circles represent the support of that Pauli operator, of range R . (b) After coarse graining by the largest such range R , every Pauli operator considered in (a) acts only on the direct neighborhood of the two sites (or single site). We call these Pauli operators hopping operators, since they have the effect of moving a topological charge from one site to a neighboring site (or on the same site).

get the desired result: a new lattice on which topological charges are represented identically at every site.

3.1.2. Hopping operators and strings. The next step is to construct hopping operators on this uniform lattice. Given two excitation configurations $\{S_a^\alpha\}$ and $\{S_b^{\alpha+x}\}$ on adjacent sites (or on the same site) and with the same charge, we can choose a Pauli operator with syndrome $\{S_a^\alpha\} \Delta \{S_b^{\alpha+x}\}$; see figure 2(a). Since there is a finite number of possible choices for $\{S_a^\alpha\}$ and $\{S_b^{\alpha+x}\}$, and two geometries of adjacency (horizontal and vertical), after a suitable coarse graining there will always be such an operator with support only at these two sites plus those surrounding them; see figure 2(b).

Hopping operators can be linked into string operators. Given two excitation configurations $\{S_a^\alpha\}$ and $\{S_b^\beta\}$, with the same charge c and located on sites α and β that form the endpoints of a path γ , there exists a Pauli operator p with syndrome $\{S_a^\alpha\} \Delta \{S_b^\beta\}$ and with support restricted to the immediate neighborhood of γ ; see figure 3. We call p a string operator with charge c and endpoints $\{S_a^\alpha\}$ and $\{S_b^\beta\}$. The string operator only has excitations at its endpoints. Indeed, begin by joining hopping operators of charge c to create the string operator. Consider the site that is at the junction of two such hopping operators. The hopping operators will create two charges c at that site, resulting in a trivial charge, but may have a non-trivial syndrome. However, by the previous paragraph, there exists a Pauli operator on that site with the resulting syndrome. By including this Pauli operator in the construction of the string operator, we obtain the desired result, see figure 4.

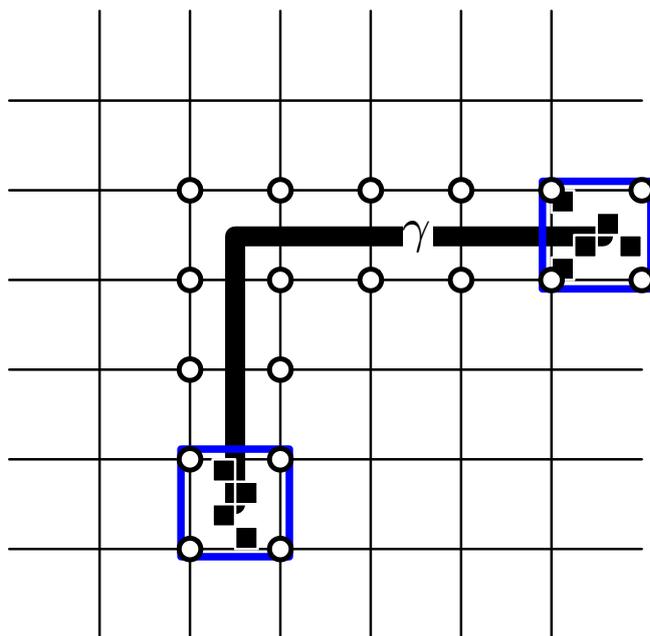


Figure 3. Example of a string operator between two excitation configurations of same charge obtained by combining hopping operators along the path γ . It acts only on the direct neighborhood of γ .

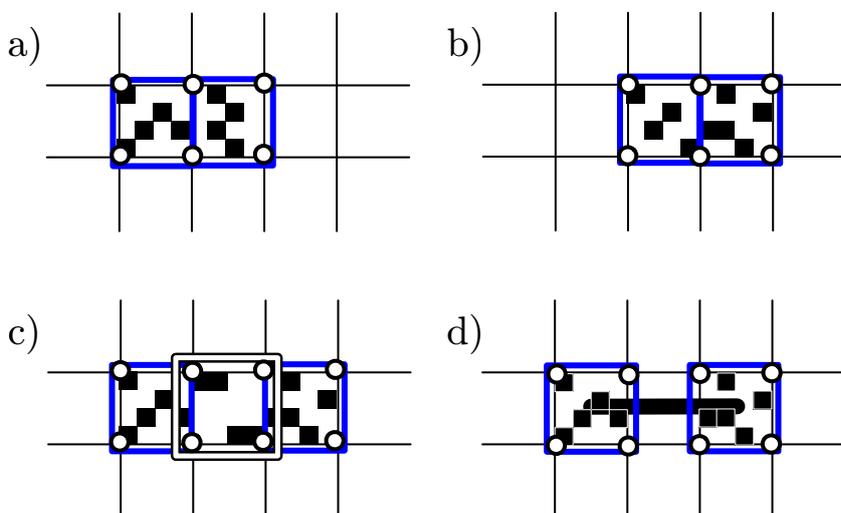


Figure 4. (a), (b) Two hopping operators of same charge overlapping on one plaquette. (c) Product of the two operators of (a), (b). Because charges square to identity, the middle plaquette contains an excitation of trivial charge. By definition of charge, there exists an operator that can correct for this excitation (and acting only on this plaquette, see text). (d) Small string operator resulting of this procedure. Iterating yields arbitrary long string operators with arbitrary configurations (of same charge) on the endpoints.

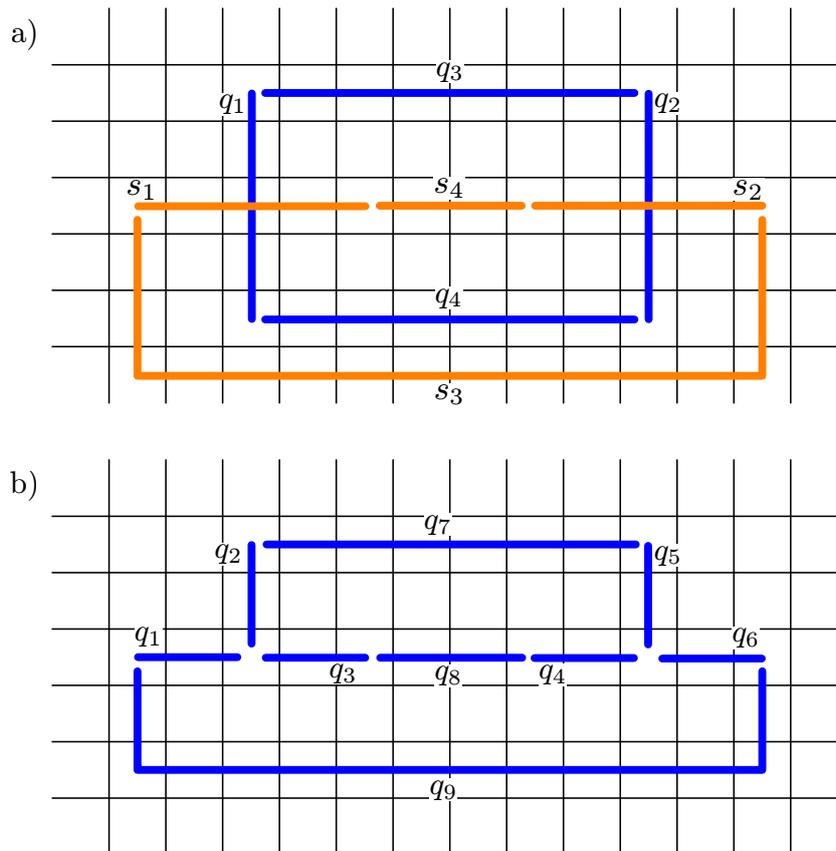


Figure 5. (a) Mutual statistics do not depend on the microscopic details of strings: only their respective charges determine the commutation relations. For two Pauli operators a and b , let $(a, b) = 1$ if a and b commute and $(a, b) = -1$ otherwise. Then, for two different choices of strings q_1, s_1 and q_2, s_2 , we must have $(q_1, s_1) = (q_2, s_2)$ because $q_1 q_2 q_3 q_4$ and $s_1 s_2 s_3 s_4$ are stabilizer operators and consequently must commute. (b) A similar reasoning holds for self-statistics: $(q_1 q_2, q_1 q_3) = (q_4 q_5, q_4 q_6)$ because $q_1 q_2 q_7 q_5 q_6 q_9$ and $q_1 q_3 q_8 q_4 q_6 q_9$ are stabilizer operators.

Anyonic statistics are recovered from string operators [33]. Namely, mutual statistics of two charges c and c' are trivial [semionic] when two crossing string operators with charges c and c' [anti]commute. Similarly, a given charge c is bosonic [fermionic] if, given three string operators q_i with charge c and with a common endpoint, the operators $q_1 q_2$ and $q_1 q_3$ [anti]commute, see figure 5—three such string operators are enough to represent a process where two identical anyons are exchanged. These commutation properties are independent of the strings chosen, as shown in figure 5; so they indeed encode the topological properties of the excitations.

3.1.3. Plaquettes. Given a non-trivial charge c and a length L , we can construct a lattice $\Lambda_c = \{\alpha_0 + nL\mathbf{x} + mL\mathbf{y} | n, m \in \mathbb{Z}\}$ of segment operators as in figure 6. Each segment is a horizontal or vertical string of charge c with common endpoints at sites separated by a distance L .

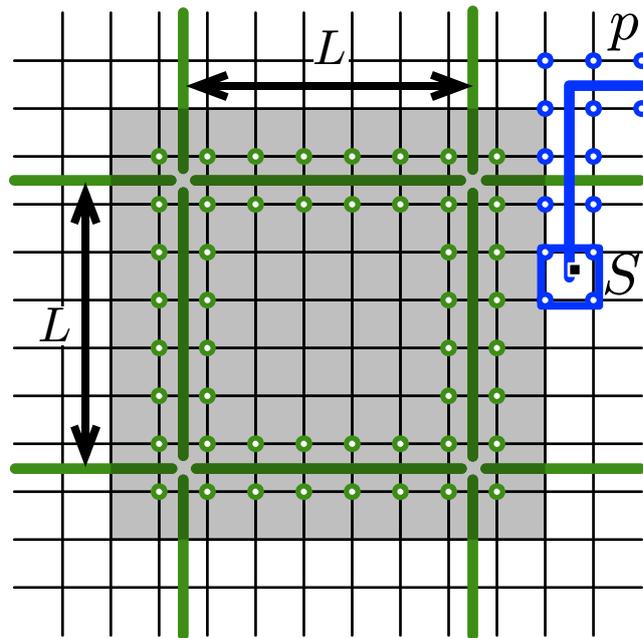


Figure 6. Lattice of horizontal and vertical segment operators of length L and of a given charge (e.g. green). Stabilizer generators of a plaquette π must be contained in the shaded region. Indeed, if there existed at least one generator outside this region (represented by a black square), then we could build a string operator anti-commuting only with this particular generator (we imagine the other endpoint to be as far as needed from the plaquette). Thus, the string must also anti-commute with the plaquette. However, the plaquette and the string do not share support (green and blue circles, respectively) and then must commute, a contradiction.

The product of the segments forming a plaquette π on Λ_c is, up to a phase, a stabilizer. This follows from the fact that it has trivial syndrome and is local, and only stabilizers can have that property by definition of TSC. Thus, such a plaquette operator π must be proportional to a product of stabilizer generators $\{S_a\}$. The support of these $\{S_a\}$ is highly constrained. Firstly, if the support of a stabilizer generator S is not contiguous to the smallest ball containing the plaquette operator π , then S cannot be one of the S_a generating π . To prove this, we can construct a string p as in figure 6 with one endpoint S and the other endpoint as far away as needed so that it does not overlap with any of the S_a . This p commutes with π because they do not share support, but anti-commutes only with the stabilizer generator S (and possibly other stabilizer generators arbitrarily far), showing that S cannot be one of the generators of π . Secondly, there cannot be a hole inside the plaquette π where none of the S_a generating π has support; see figure 7. To prove this, we can construct the product q of those S_a with support in the lower half of the lattice; see figure 7. The resulting string q is not closed and is a stabilizer, so it must have trivial charge, but it coincides with p in the lower part of the plaquette, a contradiction since p is charged.

As a consequence of these geometrical constraints, we can choose in a TI way for each plaquette π^α a stabilizer generator S^α on its central area (see figure 8), that is not shared with the other plaquettes. We can substitute the stabilizer generator S^α by π^α —adjusting the phase so

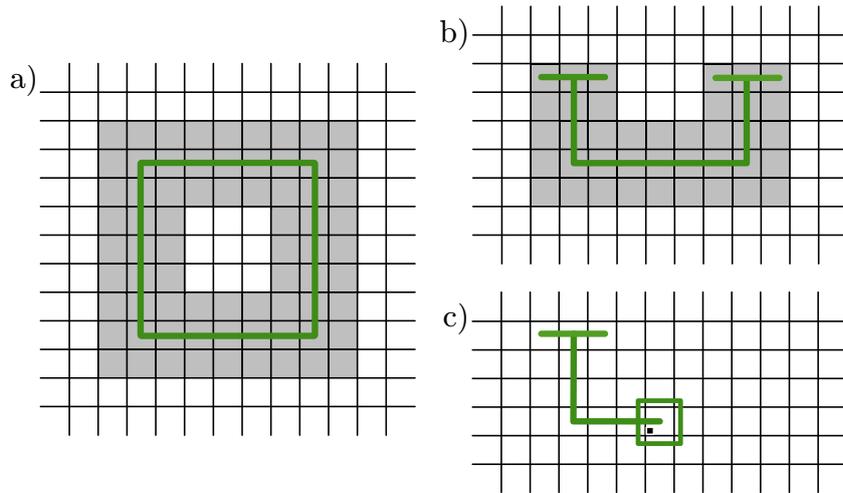


Figure 7. The shaded regions represent the union of the support of the stabilizer generators $\{S_a\}$ entering into the decomposition of π . The green loop is a string operator forming the plaquette π . (a) We consider the possibility that the support of the stabilizer generators of a plaquette has a hole inside it. (b) By multiplying the bottom half of the stabilizer generators we obtain a stabilizer operator, which locally looks like a green string on its lowest portion. (c) If we cut this operator in half vertically, its new endpoint will contain an excitation of green charge, since it is locally identical to the green string. However, the other endpoint has no excitation at all since it is constructed with stabilizers generators; so it holds a trivial charge, a contradiction.

that it is a stabilizer—to get a new set of independent LTI stabilizer generators of \mathcal{S} . Moreover, by considering a string p with an endpoint S^α and the other one away from π^α (see figure 8), we see that c (the charge of the segment operators used to construct π) and the charge of p have semionic mutual statistics.

3.1.4. Canonical charges. It follows from this last observation that the only charge that braids trivially with all other charges is the trivial charge; for all other charge c there exists at least one charge c' with which it has semionic statistics. Using this fact, we now want to organize topological charges into a canonical form. For doing this, we need to pick a set of N ‘elementary charges’ c_i that generate all other charges under fusion. Consider the $N \times N$ symmetric matrix \mathbb{S}_{ij} over \mathbf{Z}_2 whose i th diagonal element is 0 [1] if c_i is a boson [fermion] and off diagonal element ij is 0 [1] when the mutual statistics of c_i and c_j are trivial [semionic]. This matrix can be used to define the notion of a chiral model, used in the statement of our main theorem.

Definition 2. A TSC is chiral if $\text{Tr}\mathbb{S} \neq 0$.

We can transform the matrix \mathbb{S} into the canonical form $I_{N/2} \otimes \sigma_x$, assuming that the anyon model is not chiral (in which case the last two diagonal entries would be 1). Firstly, note that exchanging c_i with c_j has the obvious effect of permuting rows and columns of \mathbb{S} and that substituting c_i with $c_i c_j$ will produce a new matrix with $\mathbb{S}'_{ii} = \mathbb{S}_{ii} + \mathbb{S}_{jj} + \mathbb{S}_{ij}$, $\mathbb{S}'_{ij} = \mathbb{S}_{ij}$ and $\mathbb{S}'_{ik} = \mathbb{S}_{ik} + \mathbb{S}_{jk}$ for $k \neq i, j$ (plus symmetric equations so that \mathbb{S} remains symmetric), while other

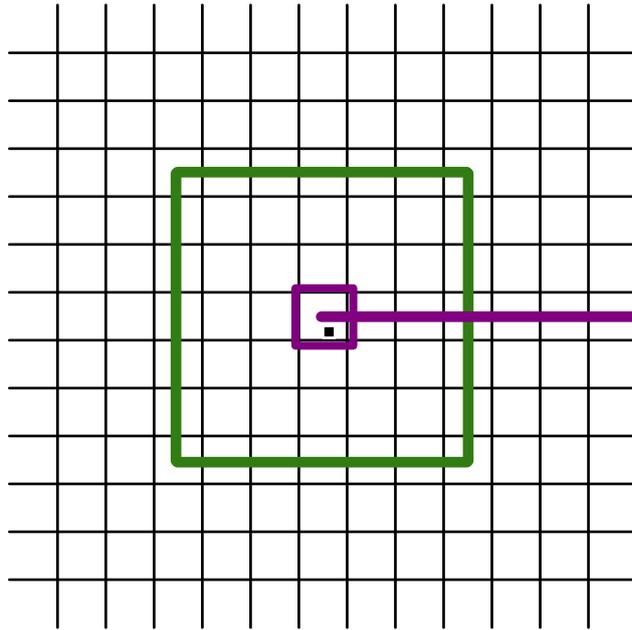


Figure 8. Firstly, we consider any of the generators, say S_i , of a plaquette, π_i , which is in its center (represented by the black square). We know such an operators exists (cf figure 7). Moreover, this stabilizer generator is not involved in any other plaquette due to its location far from the boundary (cf figure 6). We can then substitute S_i and π_i in our stabilizer generator set and we do this for every plaquette in a TI way. Secondly, we consider a string operator which anti-commutes only with the stabilizer generator S_i (we put the other end arbitrary far). It follows that this string operator anti-commutes with π_i which in turn implies that it anti-commutes with the string segment it crosses. This means the two charges (green and purple) have semionic interaction.

elements remain the same. As a first step, permute the c_i -s such that $S_{12} \neq 0$, which is always possible. We then perform a Gaussian elimination. For $i > 2$ perform the substitutions $c_i \rightarrow c_i c_1^{S_{2i}} c_2^{S_{1i}}$ so that $S_{1i} = S_{2i} = 0$. Repeating this $N/2 - 1$ more times we get $S = I_{N/2} \otimes \sigma_x + \mathbb{D}$ with \mathbb{D} a diagonal matrix with entries \mathbb{D}_i . If $\mathbb{D}_{2i-1} = 0$ and $\mathbb{D}_{2i} = 1$ substitute $c_{2i} \rightarrow c_{2i-1} c_{2i}$, and similarly for $\mathbb{D}_{2i-1} = 1$ and $\mathbb{D}_{2i} = 0$, so that we get $\mathbb{D}_{2i-1} = \mathbb{D}_{2i} = 0$. Then, there must be an even number of indices i such that $\mathbb{D}_{2i-1} = \mathbb{D}_{2i}$ (otherwise the model would be chiral). Pick any pair i, j of such indices and substitute

$$c_{2i-1} \rightarrow c_{2i-1} c_{2j}, \quad (3)$$

$$c_{2i} \rightarrow c_{2i-1} c_{2i} c_{2j}, \quad (4)$$

$$c_{2j-1} \rightarrow c_{2j-1} c_{2i}, \quad (5)$$

$$c_{2j} \rightarrow c_{2j-1} c_{2j} c_{2i}. \quad (6)$$

Repeating this procedure we arrive finally to $\mathbb{D} = 0$ as desired. Thus, we obtain a set of N canonical generating charges e^i and m^i , $i = 1, \dots, N/2$, that interact topologically as if they were the electric and magnetic charges of $N/2$ copies of KTC.

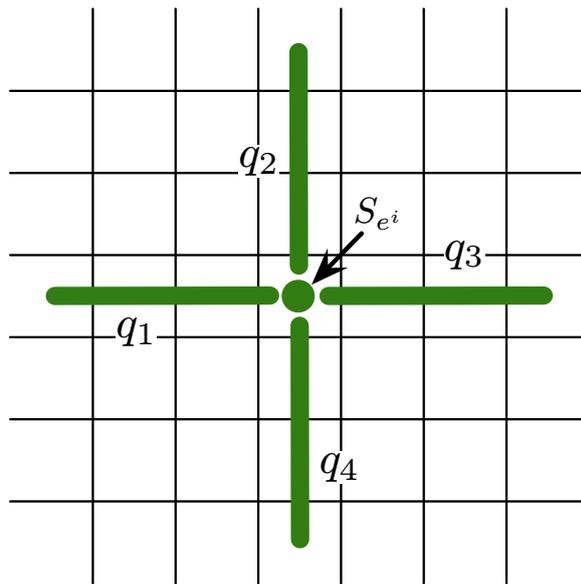


Figure 9. We consider the meeting points of the segment operators of the lattice used to build plaquettes earlier (cf figure 6). We can ensure that these string operators commute with one another. Take any stabilizer generator, S_{e^i} , that is contained in the meeting point and that has the same charge as the segments, say e^i . Then if $(q_1, q_2) = -1$, we multiply q_1 by S_{e^i} such that $(S_{e^i} q_1, q_2) = 1$. The bosonic character of e^i ensures that the remaining commutations among the q_i are as desired, e.g. $(S_{e^i} q_1 q_2, S_{e^i} q_1 q_3) = 1$ implies $(S_{e^i} q_1, q_3) = 1$.

Since every site of the lattice contains an excitation of every topological charge, we can naturally identify these canonical elementary charges e^i and m^i with stabilizer operators at a given site, in a TI way. Furthermore, we can change the generators of \mathcal{S} in a TI way to include these stabilizer operators associated to elementary charges. Just notice that if S and $\{S_i\}$ are stabilizer generators with charges c and c_i , substituting each S_i with $S'_i = SS_i$ will give rise to a new set of independent stabilizer generators where S has charge $c \prod_i c_i$ and S'_i has charge c_i . Thus, to every elementary charge c , we can associate stabilizer generators S_c^α in a TI way.

Take any canonical electric charge e^i and choose a stabilizer generator S_{e^i} of charge e^i . Construct a TI lattice of segment operators with common endpoints at S_{e^i} and its translations. There are four segments meeting at S_{e^i} , call them q_i , $i = 1, \dots, 4$; see figure 9. If q_i does not commute with q_1 , substitute it with $S_{e^i} q_i$, and do this at the other endpoints in a TI way. Thanks to the bosonic character of e_i , the new segment operators all commute with each other. We can adjust their phases so that they are Hermitian. Then, plaquette operators π constructed from the lattice of segment operators are either stabilizer or stabilizers with a negative sign. In the second case, we can negate those horizontal segment operators at every other line, which takes us back to the first case. Now choose one of these plaquette stabilizers π , a stabilizer generator S_{m^i} with charge m^i and a string p with an endpoint m^i and the other one far away, as in figure 8. Then p anti-commutes with π and S_{m^i} , showing that S_{m^i} is one of the generators of π . Then we can proceed as above and attach, in a TI way, an exclusive generator $S_{m^i}^\alpha$ with charge m^i to each plaquette π^α . As above, we can substitute $S_{m^i}^\alpha$ with π^α to get a new set of independent

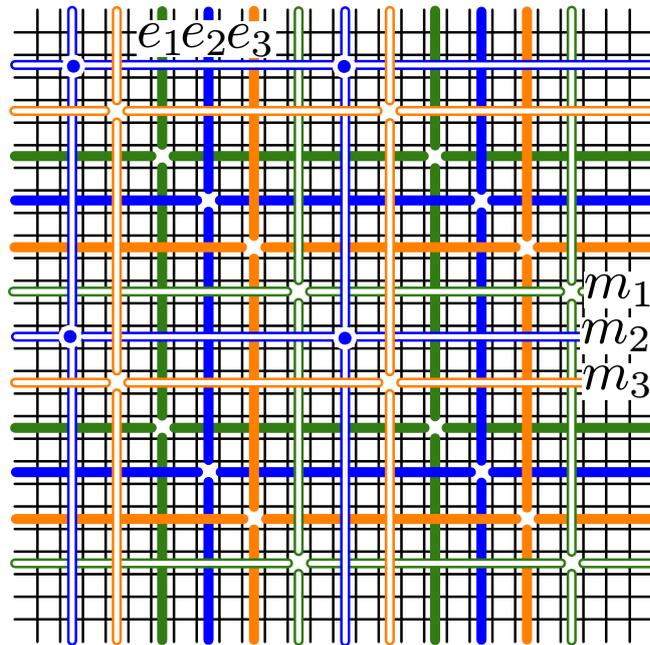


Figure 10. $2N$ lattices of segment operators, where N is the number of canonical pairs of charges (here $N = 3$). The respective offsets are chosen to insure that strings of different charges cross properly (if they do overlap) such that their commutation relations are well defined. A stabilizer generator of each plaquette can be chosen to be replaced in the stabilizer generator set. Here, an example is given for the e^2 lattice (blue): the blue circles represent stabilizer generators of charge S_{m^2} .

LTI stabilizer generators. Clearly each π^α has charge m^i . In this argument we can of course exchange e^i and m^i .

3.1.5. Canonical stabilizer generators and mapping. We can put together N such lattices, one per canonical charge generator, as in figure 10. We refer to the lattice constructed from e^i (m^i) segment operators as the e^i (m^i) lattice. This particular geometry guarantees that the commutation relations of segment operators only depend on their charge. Note that a plaquette π in the e^i lattice contains a single vertex of the m^i lattice, so that we can attach to π the corresponding stabilizer generator S_{m^i} with charge m^i that lies at that vertex. Again this works just as well exchanging e_i and m_i . It follows that we can replace each stabilizer generator that is the endpoint of segment operators of one of the N lattices with the enclosing plaquette of the same charge (see figure 10), obtaining a new set of independent LTI stabilizer generators.

This new set of stabilizer generators breaks into two disjoint subsets. The first subset of stabilizer generators are the plaquette operators constructed in the previous paragraph. The second subset contains all the other stabilizer generators. These generators commute with segment operators and have trivial charge. To prove this second statement, let S be a stabilizer generator of the second kind. Then there exists a number of plaquette operators $\{\pi_a\}$ with the same total charge as S and a Pauli operator p with syndrome $\{S\} \cup \{\pi_a\}$ (this is true because plaquette operators generate all charges by construction). However, every Pauli operator p

anti-commutes with an even number of plaquettes of a given charge. The reason for this is much like in KTC. If p anti-commutes with a given segment operator, then the two plaquette operators that share this segment operator are affected, so excitations of any given elementary charge always appear in pairs. It follows that the charge of $\{\pi_i\}$, and thus of S , is trivial.

The segment operators form a canonical basis for a subgroup \mathcal{P}_S of Pauli operators. Every e^i -segment is crossed by a unique m^i -segment, so they form a canonical pair. In addition, all other pairs of segment operators commute thanks to the canonical form of the \mathbb{S} matrix. Thus, any given Pauli operator p factorizes as $p = p_1 p_2$ (uniquely up to a sign), with p_1 in \mathcal{P}_S and p_2 in its centralizer $\mathcal{Z}(\mathcal{P}_S)$.

For any stabilizer generator S that is not a plaquette formed of segment operators, as in figure 10, there exists a Pauli operator \tilde{S} that anti-commutes with S and with no other stabilizer generator, simply because the charge of S is trivial. Due to the previous point, we can choose $\tilde{S} \in \mathcal{Z}(\mathcal{P}_S)$ while preserving locality and TI. Indeed, decompose $\tilde{S} = \tilde{S}_1 \tilde{S}_2$ with $\tilde{S}_1 \in \mathcal{P}_S$ and $\tilde{S}_2 \in \mathcal{Z}(\mathcal{P}_S)$ and substitute \tilde{S} with \tilde{S}_2 .

Label all the segment operators σ_c^α by their charge $c = e^i$ or m^i and lattice location $\alpha \in \mathbb{Z}^2$ (lattice sites are now located at the crossing of segment operators, so it is tilted at 45°). Similarly, label all the non-plaquette stabilizer generators S_a^α , where a labels the different stabilizers at a given lattice site. We can arrange the S_a^α and \tilde{S}_a^α in a canonical form. Choose an ordering in \mathbb{Z}^2 that is TI, so that $\alpha < \beta$ iff $\alpha + \gamma < \beta + \gamma$, e.g. $(\alpha_x, \alpha_y) < (\beta_x, \beta_y) \Leftrightarrow \alpha_y < \beta_y$ or $\alpha_y = \beta_y$ and $\alpha_x < \beta_x$. Extend that ordering on pairs (α, a) , for instance $(\alpha, a) < (\beta, b) \Leftrightarrow \alpha < \beta$ or $\alpha = \beta$ and $a < b$. For each \tilde{S}_a^α , consider the finite set

$$B(\tilde{S}_a^\alpha) := \{(\beta \in \mathbb{Z}^2, b) | (\beta, b) < (\alpha, a), [\tilde{S}_a^\alpha, \tilde{S}_b^\beta] \neq 0\}$$

and perform the substitution

$$\tilde{S}_a^\alpha \leftarrow \tilde{S}_a^\alpha \prod_{(\beta, b) \in B(\tilde{S}_a^\alpha)} S_b^\beta.$$

Then the \tilde{S}_a^α are TI and together with the S_a^α form a canonical Pauli basis of $\mathcal{Z}(\mathcal{P}_S)$.

We have thus constructed a local TI canonical basis for the Pauli group consisting of pairs of segment operators $(\sigma_{e^i}^\alpha, \sigma_{m^i}^\alpha)$ and pairs $(S_a^\alpha, \tilde{S}_a^\alpha)$ consisting of stabilizer generators with trivial charge and their canonically conjugated partner. The mapping to KTC is now trivial. The $(\sigma_{e^i}^\alpha, \sigma_{m^i}^\alpha)$ map to the single-qubit Pauli operators $(\sigma^{z, \alpha, i}, \sigma^{x, \alpha, i})$ where i labels the $\frac{N}{2}$ distinct copies of KTC and α labels the (tilted) lattice sites. We obtain the usual picture of qubits located on edges of the lattice by choosing a lattice rotated by 45° . The additional Pauli operators $(S_a^\alpha, \tilde{S}_a^\alpha)$ are mapped onto auxiliary qubits $(\sigma^{z, \alpha, a}, \sigma^{x, \alpha, a})$. The constraint $S_a^\alpha = +1$ implies that these auxiliary qubits are all in the state $|0\rangle$.

3.2. Example

We illustrate this mapping for TCCs [15, 25]. A TCC can be constructed on any three-valent lattice with three-colorable faces, but we take in particular the square–octagon regular lattice of figure 11. This lattice is particularly useful in terms of fault-tolerance [15]. Qubits are located at the vertices of the lattice, and there are two stabilizer operators per plaquette p

$$S_p^\sigma = \bigotimes_{e \in E_p} \sigma^e, \quad \text{with } \sigma \in \{\sigma_x, \sigma_z\}. \quad (7)$$

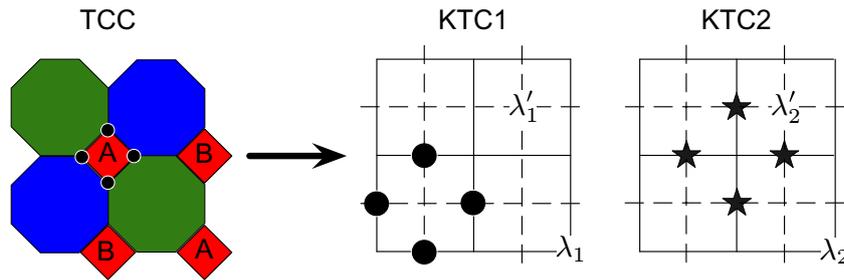


Figure 11. Regular square–octagon lattice for TCC. The diamonds can be labeled A or B according to a chessboard pattern. There are two stabilizers equation (7) associated with each plaquette. Here is an example of the mapping from one TCC to two copies of KTC. The black dots (stars) represent σ_z (σ_x) operators. A Z-plaquette on a A-diamond of the TCC gets mapped to a plaquette operator on the first KTC and to a site operator on the second KTC. The complete mapping for one-qubit Pauli operators is shown in figure 12.

The excitations in this model carry 16 different topological charges that correspond exactly to the charges obtained from two copies of KTC. For the two copies of KTC, these 16 charges are generated by the four ‘elementary’ charges e^j and m^j with $j = 1, 2$ labeling the two KTCs. Among the 16 charges of the color code, we can choose four with the same topological interactions as the e^j, m^j . Furthermore, we can find ‘hopping operators’ for these elementary charges. In KTC, the hopping operator for, say, the charge e^1 is a σ_z operator on the first copy of KTC, as this operator has the effect of moving an e^1 charge around. Once these elementary hopping operators have been identified, the mapping proceeds by identifying the hopping operators of the color code with those of the two KTCs. This procedure leads to the mapping, as shown in figure 12. It can be directly verified that it maps stabilizer generators of TCC to stabilizer generators of two KTCs, in this case with no need to add or remove trivial stabilizers.

4. Extension to subsystem codes

Subsystem stabilizer codes form a more general class of stabilizer codes [13, 14]. They can be defined as a pair $(\mathcal{S}, \mathcal{G})$, where \mathcal{G} is an arbitrary Pauli subgroup and \mathcal{S} a stabilizer such that $\mathcal{S} \propto \mathcal{Z}(\mathcal{G}) \cap \mathcal{G}$. Encoding is not done on the whole subspace defined by \mathcal{S} , but rather on the subsystem where the action of \mathcal{G} is trivial. This way, errors caused by operators in \mathcal{G} do not affect encoded states. Because of this, elements of \mathcal{G} are called gauge operators.

We say that a subsystem code $(\mathcal{S}, \mathcal{G})$ is LTI if \mathcal{G} admits an LTI set of generators \mathcal{G}_b . Note that some local subsystem codes admit no local stabilizer generators, e.g. [14]. Unlike them, a topological subsystem code should have a stabilizer with a local description. In addition, local operators should not recover any encoded information. Since we do not care about the effect of gauge operators, this can be formalized as follows in an infinite lattice:

Definition 3. A topological stabilizer subsystem code (TSSC) is an LTI subsystem stabilizer code $(\mathcal{S}, \mathcal{G})$ such that $\mathcal{Z}(\mathcal{S}) \propto \mathcal{G}$.

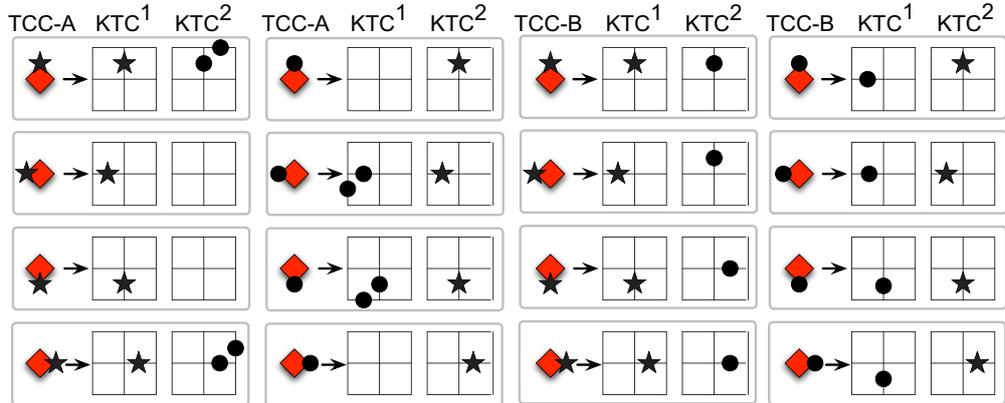


Figure 12. Mapping between the one-qubit Pauli operators of the square–octagon TCC and two copies of Kitaev’s code KTC^1 , KTC^2 . The first (last) two columns are for the A (B) sub-lattice. Circles (stars) represent σ_z (σ_x) operators. For instance, the upper left diagram indicates that a σ_x located at the top of a diamond of the A sub-lattice gets mapped to a σ_x on KTC^1 and two σ_z on KTC^2 . All commutation relations are preserved by this mapping; so it is unitary and obviously local.

There is a general strategy to understand TSSCs in terms of TSCs. Namely, to find a TSC \mathcal{S}' that lies in between the stabilizer group and the gauge group of the subsystem code, i.e. $\mathcal{S} \subset \mathcal{S}' \subset \mathcal{G}$. We can then map \mathcal{S}' invoking Theorem 1, which shows that the stabilizer generators of \mathcal{S} are locally equivalent to a subset of the stabilizer generators of several copies of KTCs. The simplest way to understand this result is to work it in reverse. Imagine starting with two copies of KTC. There are 16 topological charges, that are generated by the ‘elementary particles’ e^j and m^j , with $j = 1, 2$ labeling the two KTCs. To obtain a subsystem code, one could choose to encode information only in KTC^1 , and not enforce the stabilizers of KTC^2 . The charges e^2 and m^2 would therefore be gapless and the associated encoded qubit would carry random information. We say that we have ‘gauged out’ the elementary charges e^2 and m^2 , and retained e^1 and m^1 as ‘proper charges’. Note that the proper charges do not topologically interact with the gauge charges, which ensures that the information they encode is protected. In this example, \mathcal{S} would be the stabilizer of KTC^1 , \mathcal{S}' would be the stabilizer of the two KTCs, and \mathcal{G} would be all the operators acting on KTC^2 and the stabilizers of KTC^1 .

A slightly less trivial example can be constructed by choosing a different set of elementary charges. Consider the four fermions $\xi_1 = m^1 \times f^2$, $\xi_2 = e^1 \times f^2$, $\xi_3 = f^1 \times m^2$, and $\xi_4 = f^1 \times e^2$ that generate all the topological charges of the two KTCs. Note that the pairs (ξ_1, ξ_2) and (ξ_3, ξ_4) are canonical in the sense that mutual statistics are semionic in a pair and trivial between particles from distinct pairs. Thus, we can choose to gauge out ξ_3 and ξ_4 and retain ξ_1 and ξ_2 as proper charges to encode information. It is interesting to note that the proper charges here form a chiral anyon model. Our main result shows that whenever a TSC \mathcal{S}' exists with $\mathcal{S} \subset \mathcal{S}' \subset \mathcal{G}$, the corresponding TSSC $(\mathcal{S}, \mathcal{G})$ can be generated this way, starting with n -copies of KTC, choosing a set of elementary charges, k of which braid trivially with the rest and are gauged out. In addition, the resulting code can be modified by a local quantum circuit.

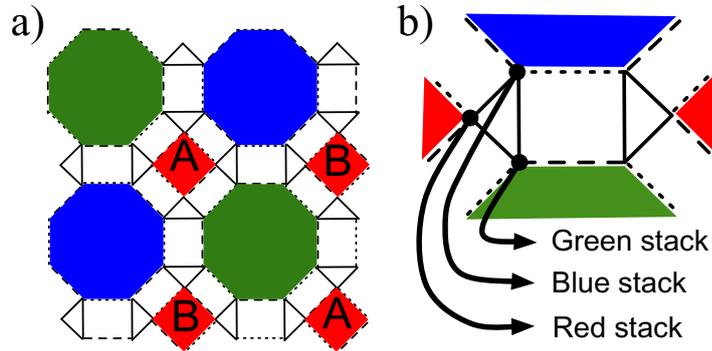


Figure 13. (a) Expanded square–octagon lattice for TSSC. Starting with the lattice of a TCC (see figure 11), each vertex is expanded into a triangle. There is one gauge operator equation (8) per edge. (b) Zoom of a region of the extended lattice and rearrangement of the qubit into three stacks.

Note that because \mathcal{S} is a strict subset of \mathcal{S}' , this mapping does not take the system to the ground state of the resulting KTCs; in a code state of the TSSC, all elements of \mathcal{S} take value +1 but the stabilizers added to \mathcal{S} to arrive at \mathcal{S}' can take any value, i.e. the system is in general not in a +1 eigenstate of the added stabilizer generators. These added generators can be measured, which will result in random excitations with trivial proper charge but arbitrary gauge charge. Thus, if one is interested to physically map a TSSC to KTCs, an additional step is required. These excitations can be eliminated by local transformations; simply pairing up elementary excitations in an arbitrary way and fusing each pair into the vacuum. Moreover, because these excitations correspond to gauge charges, this local transformation does not change the encoded information; the different ways of pairing the elementary excitations will only affect the gauge sector of the Hilbert space.

4.1. Examples

Let us illustrate this strategy with an important family of 2D subsystem codes [25] called TSSCs. Given the lattice of a TCC, we can inflate each vertex into a triangle, as shown in figure 13(a). Qubits are located on the vertices of this inflated lattice, and there is one gauge group generator associated with each pair of sites i, j connected by an edge

$$G_{ij} = \sigma^i \sigma^j \quad (8)$$

with $\sigma = \sigma_x, \sigma_y$, or σ_z for a dashed, dotted or solid edge, respectively. This code admits a set of local stabilizer generators, some of which involve a relatively large number of qubits (up to 24). Excitations are described by two elementary topological charges (ξ_1, ξ_2) , both fermions and with semionic mutual statistics, making it a chiral anyon model. The fusion rules are $\xi \times \xi \rightarrow 0$ and $\xi_1 \times \xi_2$ is a composite fermion. These topological properties are identical to those of the proper charges ξ_1 and ξ_2 constructed in the previous paragraph. This suggests that we should be able to find a TSC \mathcal{S}' with $\mathcal{S} \subset \mathcal{S}' \subset \mathcal{G}$ and \mathcal{S}' locally equivalent to n -copies of KTC with $n \geq 2$. We will present two different ways of obtaining \mathcal{S}' that are geometry independent (i.e. not restricted to the square–octagon lattice).

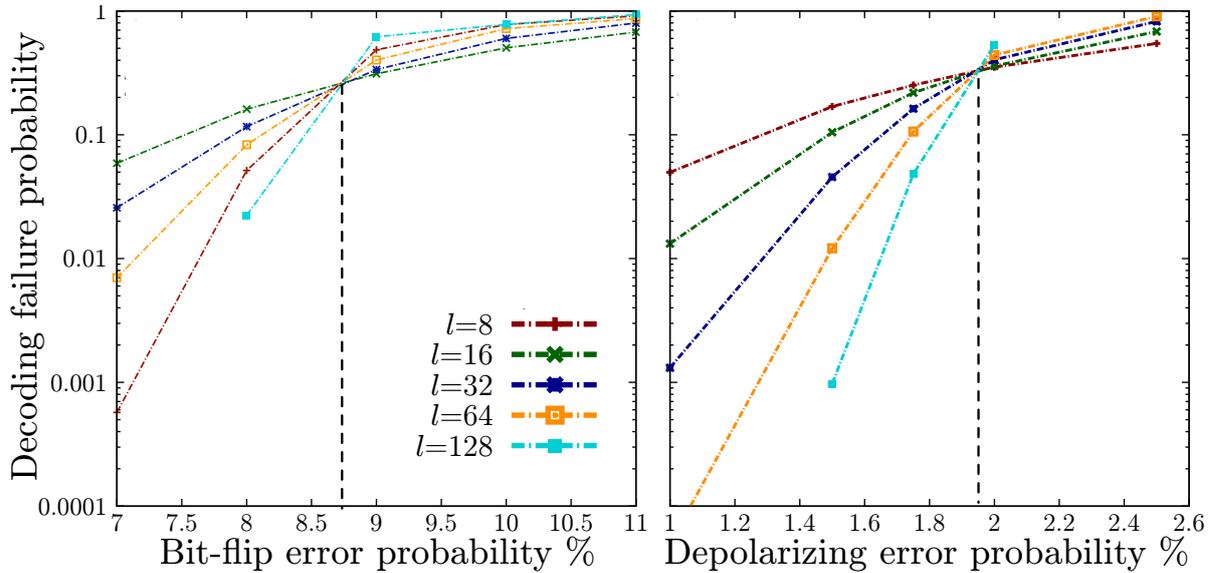


Figure 14. Decoding failure probability as a function of the error probability of each qubit for the square–octagon TCC (left) and TSCC (right), based on the algorithm of [23]. The different curves illustrate lattices of different linear size l : below a threshold probability (dotted lines), the decoding failure probability decreases with the lattice size, leading to a perfect recovery in the thermodynamic limit.

In the first construction, \mathcal{S}' is the stabilizer of three TCCs on the corresponding non-inflated lattice. Indeed, all we need to do is to rearrange the qubits. The three qubits located at the vertices of each triangle inherit the color label of the neighboring plaquette. We construct a stack of three TCC lattices—one per color—each one containing the qubits of that color; see figure 13(b). It can be easily verified that this maps the generators of \mathcal{S} to a subset of the generators of the three TCCs. We obtain \mathcal{S}' by including the other stabilizer generators of these TCCs. In the second construction, we consider the stabilizer group \mathcal{S}_z generated by the gauge operators of the form $\sigma_z \sigma_z$ (solid edges), which clearly is a subgroup of $\mathcal{Z}(\mathcal{S})$. It follows from the results in [34] that $\mathcal{S}' := \mathcal{S}\mathcal{S}_z$ is a TSC with the same topological charges as a TCC. These two constructions illustrate that the quantum dimension of the intermediate code \mathcal{S}' is not uniquely determined, since in the first case we have $D = 2^6$ and in the second $D = 2^2$.

5. Decoding

When the system is prepared in the ground state of the Hamiltonian (1) H , all stabilizers have value $+1$. However in the presence of errors, this will not be the case in general. The problem of decoding a quantum code consists in identifying the most likely recovery to restore the encoded state from partial information coming from the measurement of the stabilizer operators, whose ± 1 outcomes are called error syndrome. Not all codes can be decoded efficiently, but fast approximate algorithms have been devised for KTC. The one presented in [12] uses Edmonds' minimum matching algorithm [35] to find the shortest path that recombines all electric particles in pairs and independently all magnetic particles in pairs. For N qubits, it runs in time N^3 .

The algorithm proposed in [23] uses renormalization group approximations to find the homological class of errors with the highest probability. It runs in time $\log N$. An efficient decoder was also devised for a TSSC on a particular lattice [24], but in general each new code requires a tailored decoding technique.

Our techniques can be used to decode any 2D TSCs. Indeed, our main theorem shows the existence of a transformation that maps an error syndrome of a TSC to a syndrome on the KTCs. Given that syndrome, we can run any of the known decoding algorithms [12, 23] on each of the KTCs, and translate the proposed recovery back to the original TSC. In fact, the idea extends to any TSSCs. As discussed in the proof of our main theorem, we can always find a set of canonical elementary charges e^i, m^i (these could be fermions for TSSCs) that generate all the topological charges in the code. Then, the decoding problem boils down to matching all elementary defects in pairs, just like for KTC. The advantage here is that decoding does not require an explicit unitary mapping between the codes, but only a mapping between excitations. We have used this technique, combined with the decoding algorithm of [23], for the TCC on the square–octagon lattice of figure 11 on a bit-flip channel and found an error threshold of roughly 8.7% (see figure 14), in good agreement with the Monte Carlo estimate of 10.9% [36] for ideal error correction. We have also used this technique for the TSCC on the square–octagon lattice of figure 11 on a depolarizing channel and found an error threshold of roughly 1.95% (see figure 14), in good agreement with the estimate of 2% [24] for a closely related code in a five-square lattice.

6. Conclusion

We have demonstrated that 2D TSCs all belong to one universal topological phase by constructing an explicit local mapping onto multiple copies of KTC. This result also carries to a certain class of 2D subsystem codes, and in particular to all topological subsystem color codes. These local maps enable us to extend many properties of Kitaev’s code to all 2D codes, and in particular directly yield efficient decoding algorithms for error correction. It could also have important implications for fault-tolerant quantum computation.

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