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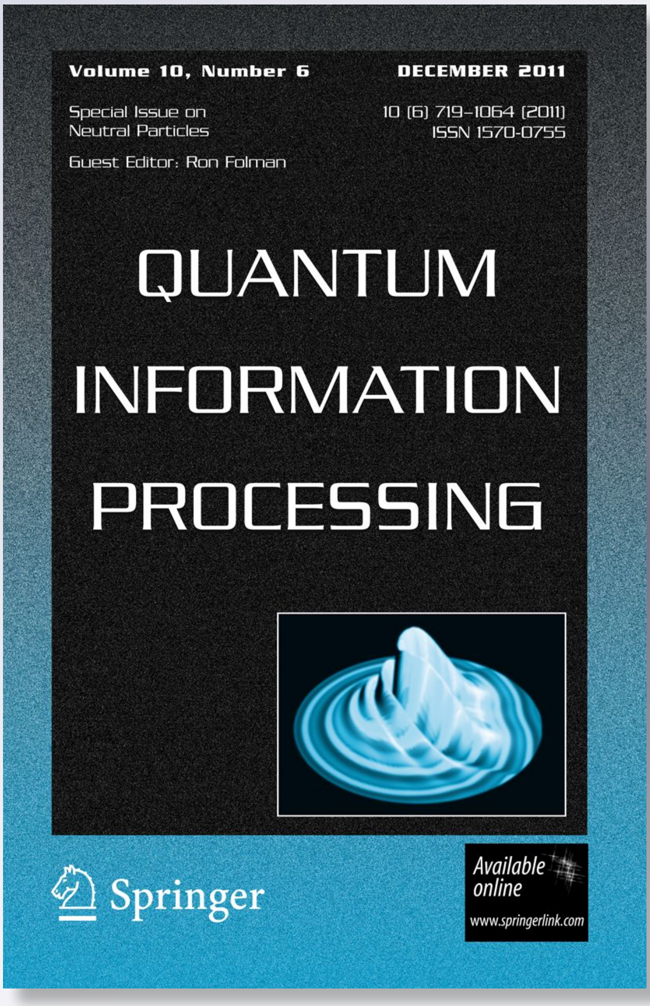

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Hitting time of quantum walks with perturbation

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Abstract The hitting time is the required minimum time for a Markov chain-based walk (classical or quantum) to reach a target state in the state space. We investigate the effect of the perturbation on the hitting time of a quantum walk. We obtain an upper bound for the perturbed quantum walk hitting time by applying Szegedy's work and the perturbation bounds with Weyl's perturbation theorem on classical matrix. Based on the definition of quantum hitting time given in MNRS algorithm, we further compute the delayed perturbed hitting time and delayed perturbed quantum hitting time (DPQHT). We show that the upper bound for DPQHT is bounded from above by the difference between the square root of the upper bound for a perturbed random walk and the square root of the lower bound for a random walk.

Keywords Markov chain · Quantum walk · Hitting time · Matrix perturbation · Random walk · Delayed perturbed quantum hitting time · Delayed perturbed hitting time

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1 Introduction

Markov chains and random walks have been useful tools in classical computation. One can use random walks to obtain the final stationary distribution of a Markov chain to sample from. In such an application the time the Markov chain takes to converge, i.e., *convergence time*, is of interest because shorter convergence time means lower cost in generating a sample. Sampling from stationary distributions of Markov chains combined with simulated annealing is the core of many clever classical approximation algorithms. For instance, approximating the volume of convex bodies [1], approximating the permanent of a non-negative matrix [2], and the partition function of statistical physics models such as the Ising model [3] and the Potts model [4]. In addition, one can also use the random walks to search for the *marked* state, in which the *hitting time* is of interest because hitting time indicates the time it requires to find the marked state.

The concept of quantum random walk was introduced by Aharonov, Daviovich and Zagury [5]. Using the spectral distribution associated with the adjacency matrix of graphs, one can calculate [6] the amplitudes of continuous-time quantum walk. Both random walk and quantum walk can be defined either in continuous or discrete time. It is shown [7] that continuous-time quantum walk can be obtained as an appropriate limit of discrete-time quantum walks. Interested readers can reference [7] for details.

In comparison to classical random walk, quantum walk provides a quadratic speed-up in hitting time. Quantum walk has been applied to solving many interesting problems [8], such as searching problems, group commutativity, element distinctness, restricted range associativity, triangle finding in a graph, and matrix product verification. For some problems that can be reformulated in terms of decision trees, it was shown [9] that to reach level n of the tree the classical random walk algorithm requires time exponential in n but the quantum algorithm succeeds in polynomial time.

In addition, perturbations of classical Markov chains are widely studied with respect to hitting time and stationary distribution. Since a quantum system is susceptible to the environmental noise, we are interested to know what effect perturbation has on currently existing quantum walk based algorithms.

This work is organized as follows. In Sect. 2 we present the deviation effect of perturbation on the spectral gap of a classical Markov chain. In Sect. 3 we discuss how the hitting time would be affected because of the perturbation. We explore the upper bounds for the perturbed hitting time quantumly and the time difference (delayed perturbed hitting time) both quantumly and classically. Finally in Sect. 4, we make our conclusion.

2 Classical spectral gap perturbation

Given a stochastic symmetric matrix $P \in \mathbb{C}^{n\times n}$, we can quantize the Markov chain [10]. Wocjan, Nagaj and one of us [11] showed that the implementation of one step of quantum walk can be achieved efficiently. However, the above settings always are under the assumption of perfect scenarios. In real life there are many sources of errors that would perturb the process. Noise might be propagated along with the input source

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or they might be introduced during the process. Here we look solely at the noise that are introduced at the beginning of the process.

The noise can be introduced due to the precision limitation and the noisy environment. For instance, not all numbers have a perfect binary representation and the approximated numbers would cause perturbation. Suppose our input decoding mechanism can always take the input matrix and represent it in a symmetric transition matrix Q, where Q can be perfectly represented and this is the matrix closest to the original matrix P that the system can prepare.

Let E be the noise that is introduced because of system's precision limitation and the environment, we can express the transition matrix as

$$Q = P + E. \tag{1}$$

Classically, much research [12–17] has focused on the spectral gaps and stationary distributions of the matrices with perturbation. In a recent work by Ipsen and Nadler [12], they refined the perturbation bounds for eigenvalues of Hermitians. Throughout the rest of the paper, $\|\cdot\|$ always denotes the l_2 norm, unless otherwise specified. Based on their result, we summarized the following:

Lemma 1 *Suppose P and $Q \in \mathbb{C}^{n\times n}$ are Hermitian symmetric transition matrices with respective eigenvalues*

$$0 < \lambda_{n-1}(P) \leq \cdots \leq \lambda_0(P) = 1, \quad 0 < \lambda_{n-1}(Q) \leq \cdots \leq \lambda_0(Q) = 1, \tag{2}$$

and $Q = P + E$, then

$$\max_{0\leq i\leq n-1} |\lambda_i(P) - \lambda_i(Q)| \leq \|E\|. \tag{3}$$

Furthermore, the spectral gap δ of P and the spectral gap Δ of Q have the following relationship

$$\delta - \|E\| \leq \Delta \leq \delta + \|E\|. \tag{4}$$

Proof Equation (3) is a direct result from the Weyl's Perturbation Theorem. The *Weyl's Perturbation Theorem* bounds the worst-case absolute error between the ith exact and the ith perturbed eigenvalues of Hermitian matrices in terms of the l_2 norm [14,15]. And since

$$1 - \lambda_1(P) = \delta, \quad 1 - \lambda_1(Q) = \Delta, \tag{5}$$

by Eq. (3) we have $|\delta - \Delta| \leq \|E\|$. Therefore, in general we can bound the perturbed spectral gap Δ as

$$\delta - \|E\| \leq \Delta \leq \delta + \|E\|. \tag{6}$$

□

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Generally speaking, the global norm of E might be very large when the dimensions $n >> 1$ [18]. However, in our case because E is the difference between two very close stochastic symmetric matrices, its global norm would never become large.

3 Hitting time of Markov chain based walks

For the purpose of being complete, we need to cite several definitions and results used in the MNRS algorithm [19] in this section. We recommend interested readers to reference [19] for details.

Let P be a reversible and ergodic transition matrix with state space Ω and positive eigenvalues. Suppose P is column-wise stochastic and $|\Omega| = n$, then let the Markov chain $(X_1, \ldots, X_n)$ under discussion have a finite state space Ω and transition matrix P.

Definition 1 For $x \in \Omega$, denote the *hitting time* for x

$$\mathrm{HT}(P, x) = \min\{t \geq 1 : X_t = x\}. \tag{7}$$

$\mathrm{HT}(P, x)$ is the expected number of transition matrix P invocations to reach the state x when started in the initial distribution π.

Definition 2 For an $n \times n$ matrix P, P_{-x} denotes the $(n-1) \times (n-1)$ matrix of P where the row and column indexed by x are deleted. For a vector v, v_{-x} is the vector that omits the x-coordinate of v. Similarly, suppose $\{M\} = \{x_1, \ldots, x_m\}$, then $P_{-\{M\}}$ denotes the $(n-m) \times (n-m)$ matrix of P where the rows and columns indexed by $x_1, x_2, \ldots,$and x_m are deleted.

Definition 3 Denote the vector space $\mathcal{H} = \mathbb{C}^{|\Omega| \times |\Omega|}$. For a state $|\psi\rangle \in \mathcal{H}$, define $\Pi_\psi = |\psi\rangle\langle\psi|$ as the orthogonal projector onto $Span(|\psi\rangle)$. Let $\mathcal{A} = Span(|y\rangle|p_y\rangle : y \in \Omega)$ be the vector subspace of $\mathcal{H}$ where

$$|p_y\rangle = \sum_{z \in \Omega} \sqrt{p_{zy}}|z\rangle. \tag{8}$$

$\mathcal{A}$ is spanned by a set of mutually orthogonal states $\{|\psi_i\rangle : i = 1, 2 \ldots, |\Omega|\}$, then let $\Pi_{\mathcal{A}} = \sum_i \Pi_{\psi_i}$. Similarly, $\mathcal{A}_{-x} = Span\{|y\rangle|p_y\rangle : y \in \Omega \backslash \{x\}\}$.

Definition 4 The unitary operation $W(P) = (S \cdot (2\Pi_{\mathcal{A}} - I))^2$ defined on $\mathcal{H}$ is the quantum analog of P. Similarly, the unitary operation $W(P, x) = (S \cdot (2\Pi_{\mathcal{A}_{-x}} - I))^2$ defined on $\mathcal{H}$ is the quantum analog of P_{-x}. S is the swap operation defined by $S|y\rangle|z\rangle = |z\rangle|y\rangle$.

Fact 1 [10,19] *Let $x \in \Omega$ and $|\mu\rangle = |x\rangle|p_x\rangle$. Let $U_2 = S(2\Pi_{\mathcal{A}} - I)$ and $U_1 = I - 2|\mu\rangle\langle\mu|$. When P is reversible, then $U_2^2 = W(P)$ and $(U_2U_1)^2 = (S(2\Pi_{\mathcal{A}_{-x}} - I))^2 = W(P, x)$.*

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Proof *Since* $\Pi_{\mathcal{A}} = \left(\sum_{y=1,y\neq x}^{|\Omega|} |y\rangle|p_y\rangle\langle y|\langle p_y|\right) + |\mu\rangle\langle\mu|$, *then we have*

$$\begin{aligned} U_2U_1 &= S(2\Pi_{\mathcal{A}} - I)(I - 2|\mu\rangle\langle\mu|) \\ &= S(2\Pi_{\mathcal{A}} - 2|\mu\rangle\langle\mu| - I) \\ &= S(2\Pi_{\mathcal{A}_{-x}} - I) \end{aligned}$$

□

3.1 Classical hitting time

By [19], the x-hitting time of P can be expressed as $\mathrm{HT}(P, x) = \pi^{\dagger}(I - P_{-x})^{-1}u_{-x}$, where u is an all-ones vector. It is known that

$$\pi_{-x}^{\dagger}(I - P_{-x})^{-1}u_z = \sqrt{\pi_{-x}}^{\dagger}(I - S_{-x})^{-1}\sqrt{\pi_{-x}} \tag{9}$$

where $S_{-x} = \sqrt{\Pi_{-x}}P_{-x}\sqrt{\Pi_{-x}}^{-1}$ with $\Pi_{-x} = diag(\pi_i)_{i\neq x}$ and $\sqrt{\pi_{-x}}$ is the entry-wise square root of π_{-x}. Let $\{v_j : j \leq n-1\}$ be the set of normalized eigenvectors of S_{-x} where the eigenvalue of v_j is $\lambda_j = \cos\theta_j$ with $0 \leq \theta_j < \pi/2$. By reordering the eigenvalues, let us assume that $1 > \lambda_1 \geq \cdots \geq \lambda_{n-1} > 0$. When $\sqrt{\pi_{-x}} = \sum_j \nu_j v_j$ is the decomposition of $\sqrt{\pi_{-x}}$ in the eigenbasis of S_{-x}, the x-hitting time satisfies

$$\mathrm{HT}(P, x) = \sum_j \frac{\nu_j^2}{1 - \lambda_j}. \tag{10}$$

In a similar manner, let $\tilde{S}_{-x} = \sqrt{\Pi_{-x}}Q_{-x}\sqrt{\Pi_{-x}}^{-1}$ for a perturbed matrix Q where $Q = P + E$. Then the x-hitting time for Q satisfies

$$\mathrm{HT}(Q, x) = \sum_j \frac{\tilde{\nu}_j^2}{1 - \tilde{\lambda}_j}, \tag{11}$$

where $\tilde{\lambda}_j$ are the eigenvalues of $\tilde{S}_{-x}$ and $\sqrt{\pi_{-x}} = \sum_j \tilde{\nu}_j\tilde{v}_j$ is the decomposition of $\sqrt{\pi_{-x}}$ in the eigenbasis, $\{\tilde{v}_j\}$, of $\tilde{S}_{-x}$.

Three simple facts can be observed from above description of classical hitting time.

Fact 2 S_{-x} *and* P_{-x} *are similar, they have the same eigenvalues.*

Fact 3 $\tilde{S}_{-x}$ *and* Q_{-x} *are similar, they have the same eigenvalues.*

Fact 4 *Since the entries of distribution* π *sum up to 1 , i.e.* $\sum_i(\pi_i) = 1$, *then it is obvious that* $\sqrt{\pi_{-x}}^{\dagger}\sqrt{\pi_{-x}} = \sum_i(\pi_i)_{i\neq x} \leq 1$. *Hence we know that* $\sum_i \nu_i^2 = \sum_i \tilde{\nu}_i^2 \leq 1$.

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3.2 Delayed perturbed hitting time

In this subsection, we define the delayed perturbed hitting time and its upper bound as the following.

Proposition 1 *For a Markov transition matrix P with state space Ω and limiting distribution π. Assume $|\Omega| = n$ and let $|v_i\rangle$ be the eigenvector with corresponding eigenvalue λ_i of P_{-x}. Suppose the eigenvalues of P_{-x} are ordered such that $1 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > 0$. The x-hitting time satisfies*

$$\mathrm{HT}(P, x) = \sum_{j=1} \frac{\nu_j^2}{1 - \lambda_j} \tag{12}$$

*where $\sqrt{\pi_{-x}} = \sum_{j=1}^{|\Omega|-1} \nu_j |v_i\rangle$. When given a perturbed matrix Q where $\|Q - P\| \leq \|E\|$, denote the Delayed Perturbed Hitting Time (*DPHT*(P,Q,x)) that*

$$\mathrm{DPHT}(P, Q, x) = \mathrm{HT}(Q, x) - \mathrm{HT}(P, x).$$

DPHT(P, Q, x) *can be bounded from above by*

$$\frac{1}{1 - \lambda_1 - \|E\|} - \frac{1}{1 - \lambda_1 + \gamma} \tag{13}$$

where $\lambda_1 - \lambda_{n-1} = \gamma$.

Proof Let the eigenvalues of Q_{-x} be $\tilde{\lambda}_i$. By the fact $\|Q - P\| \leq \|E\|$ and Weyl's perturbation theorem, we know that $\|Q_{-x} - P_{-x}\| \leq \|E\|$ and $|\lambda_i - \tilde{\lambda}_i| \leq \|E\|$. The delayed hitting time due to perturbation is thus

$$\begin{aligned}
\mathrm{DPHT}(P, Q, x) &= \mathrm{HT}(Q, x) - \mathrm{HT}(P, x) \\
&= \sum_{i \in \Omega} \Big(\frac{\tilde{\nu}_i^2}{1 - \tilde{\lambda}_i} - \frac{\nu_i^2}{1 - \lambda_i} \Big) \\
&\leq \Big(\sum_{i \in \Omega} \frac{\tilde{\nu}_i^2}{1 - \lambda_1 - \|E\|} \Big) - \Big(\sum_{i \in \Omega} \frac{\nu_i^2}{1 - \lambda_{n-1}} \Big) \\
&\leq \Big(\frac{1}{1 - \lambda_1 - \|E\|} - \frac{1}{1 - \lambda_1 + \gamma} \Big),
\end{aligned} \tag{14}$$

□

the last inequality is a result from Fact 4.

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3.3 Upper bound for perturbed quantum hitting time

Given two Hermitian stochastic matrices, P and Q, we explore the difference between walk operators, $W(P)$ and $W(Q)$, with respect to their hitting time. Denote the set of marked elements as $|M|$. Based on the result from Lemma 1, we have the following:

Lemma 2 *Given two symmetric reversible ergodic transition matrices P and $Q \in \mathbb{C}^{n\times n}$, where $Q = P + E$, let $W(P)$ and $W(Q)$ be quantum walks based on P and Q, respectively. Let M be the set of marked elements in the state space. Denote* $\mathrm{QHT}(P)$ *as the hitting time of walk $W(P)$ and* $\mathrm{QHT}(Q)$ *as the hitting time of walk $W(Q)$. Suppose $|M| = \epsilon N$. If the second largest eigenvalues of P and Q are at most $1-\delta$ and $1-\Delta$, respectively, then in general*

$$\mathrm{QHT}(P) = O\Bigg(\sqrt{\frac{1}{\delta\epsilon}}\Bigg), \quad \mathrm{QHT}(Q) = O\Bigg(\sqrt{\frac{1}{(\delta - \|E\|)\epsilon}}\Bigg) \tag{15}$$

where $\delta - \|E\| \le \Delta \le \delta + \|E\|$.

Proof Suppose the Markov chain P, Q and matrix E are in the following block structure

$$P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}, \quad E = \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix} \tag{16}$$

where we order the elements such that the marked ones come last, i.e., P_4, Q_4 and $E_4 \in \mathbb{C}_{|M|\times|M|}$. The corresponding modified Markov chains [10] would be

$$\tilde{Q} = \begin{pmatrix} Q_1 & 0 \\ Q_3 & I \end{pmatrix} = \begin{pmatrix} P_1 + E_1 & 0 \\ P_3 + E_3 & I \end{pmatrix}. \tag{17}$$

By [10], we have $\mathrm{QHT}(P) = O(\sqrt{\frac{1}{1-\|P_1\|}})$ and $\mathrm{QHT}(Q) = O(\sqrt{\frac{1}{1-\|Q_1\|}})$. Since we know

$$\|P_1\| \le 1 - \frac{\delta\epsilon}{2} \quad \text{and} \quad \|Q_1\| \le 1 - \frac{\Delta\epsilon}{2} \tag{18}$$

by [10] and by Cauchy's interlacing theorem we have $\|E\| \ge \|E_1\|$ [20, Cor.III.1.5], we then obtain

$$\|Q_1\| \le \min\left\{\|P_1\| + \|E\|, 1 - \frac{(\delta - \|E\|)\epsilon}{2}\right\} \tag{19}$$

as $\delta - \|E\| \le \Delta \le \delta + \|E\|$. Therefore, the hitting times for P and Q are derived. □

From the corollary above, it is clear that the noise increases the quantum hitting time. By a simple comparison with the classical hitting time, we have the following fact.

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Fact 5 *When given a perturbed quantum walk $W(Q)$, where the magnitude of noise is $\|E\|$, the quadratic speed-up gained from the quantum walk will be annihilated when $\|E\| \geq \Omega(\delta(1-\delta\epsilon))$.*

Proof Given $\|E\| = \delta(1-\delta\epsilon)$, then by Lemma 2 we have $\mathrm{QHT}(Q) = O(\sqrt{\frac{1}{\delta(1-(1-\delta\epsilon))\epsilon}}) = O(\frac{1}{\delta\epsilon})$. □

3.4 Quantum hitting time based on MNRS algorithm

Let $U = U_2U_1$ be a unitary matrix with real entries. Let $|\mu\rangle$ (see Fact 1) be the marked element where $U_1 = I - 2|\mu\rangle\langle\mu|$ and U_2 is a real unitary matrix with a unique 1-eigenvalue $|\phi\rangle$. Similar to the classical case, let $|\phi\rangle_{-\mu} = |\phi\rangle - \langle\phi|\mu\rangle|\mu\rangle$.

The potential eigenvalues for U are then ± 1 and conjugate complex numbers $(e^{i\alpha_j}, e^{-i\alpha_j})$. Let $|\phi\rangle_{-\mu}$ be the input state for the phase estimation of U, then $|\phi\rangle_{-\mu}$ can be uniquely decomposed in the eigenbasis of U as

$$|\phi\rangle_{-\mu} = \delta_0|\omega_0\rangle + \sum_j \delta_j|\omega_j^{\pm}\rangle + \delta_{-1}|\omega_{-1}\rangle \tag{20}$$

where $U|\omega_0\rangle = |\omega_0\rangle, U|\omega_{-1}\rangle = -|\omega_{-1}\rangle$ and $U|\omega_j\rangle = e^{\pm i\alpha_j}|\omega_j\rangle$. Let QH be the random variable which takes the value $1/\alpha_j$ with probability δ_j^2 and the value $1/\pi$ with probability δ_{-1}^2.

Definition 5 [19] The quantum $|\mu\rangle$-hitting time of U_2 is the expectation of QH, that is

$$\mathrm{QHT}(U_2, |\mu\rangle) = 2\sum_i \frac{\delta_j^2}{\alpha_j} + \frac{\delta_{-1}^2}{\pi}. \tag{21}$$

Hence, in order to compute the quantum hitting time of U_2, it is important to compute the spectral decomposition of U. It is shown in the following theorem.

Theorem 1 [10] *Fix an $n \times n$ column-wise stochastic matrix $\tilde{P}$,[1] and let $\{|\lambda\rangle\}$ denote a complete set of orthonormal eigenvectors of the $n \times n$ matrix D with entries $D_{jk} = \sqrt{\tilde{P}_{jk}\tilde{P}_{kj}}$ with eigenvalue $\{\lambda\}$. Then the eigenvalues of the discrete-time quantum walk $U = S(2\Pi_{\mathcal{A}} - I)$ corresponding to $\tilde{P}$ are ± 1 and $\lambda \pm i\sqrt{1-\lambda^2} = e^{\pm i arccos \lambda}$.[2]*

Let the subset M be the set of marked elements that we are searching for. The discrete-time quantum walk $U_{-\{M\}} = S(2\Pi_{\mathcal{A}_{-\{M\}}} - I)$ satisfies the above theorem

1 $\tilde{P}$ is the modified stochastic matrix of P defined in Eq. 22

2 Eigenvalues of $\tilde{D}$ in Eq. 23 are exactly the eigenvalues of $\tilde{P}_{-\{M\}}$ and eigenvalue 1

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when we modify the original transition matrix P into $\tilde{P}$ in the following manner:

$$\tilde{P}_{jk} = \begin{cases} 1 & k \in M \quad \text{and} \quad j = k \\ 0 & k \in M \quad \text{and} \quad j \neq k \\ P_{jk} & k \notin M \end{cases}$$

We can view $\tilde{P}$ in block structure as follows:

$$P = \begin{pmatrix} P_{-\{M\}} & P_2 \\ P_3 & P_4 \end{pmatrix} \longrightarrow \quad \tilde{P} = \begin{pmatrix} P_{-\{M\}} & 0 \\ P_3 & I \end{pmatrix}, \tag{22}$$

then the corresponding discriminant matrix $\tilde{D}$ is

$$\tilde{D} = \begin{pmatrix} P_{-\{M\}} & 0 \\ 0 & I \end{pmatrix}. \tag{23}$$

Fact 6 *Now let us set $M = \{x\}$. Then ± 1 and $e^{\pm i\alpha_j}$ are eigenvalues of U_{-x} where λ_j are the eigenvalues of P_{-x}. Since $\lambda_j = \cos\theta_j$ (see* Sect. 3.1*), and by use of* Theorem 1, *we know that $\theta_j = \alpha_j$.*

Furthermore, by Fact 1 we know the unitary $W(P, x) = U_{-x}^2$. The eigenvectors of U_{-x} remain the eigenvectors of $W(P, x)$ but the eigenvalues of $W(P, x)$ would be $e^{2i\alpha_j}$. Given $|\phi\rangle_{-\mu}$ as the input state, we run phase estimation of $W(P, x)$ and the corresponding quantum hitting time would be

$$\text{QHT}(P, x) = 2\sum_{j=1}^{n-1} \frac{\delta_j^2}{2\alpha_j} = \sum_{j=1}^{n-1} \frac{\delta_j^2}{\theta_j}, \tag{24}$$

the term $\frac{\delta_{-1}^2}{\pi}$ in Definition 5 disappears because the corresponding eigenphase becomes 0.

3.5 Delayed perturbed quantum hitting time

In this subsection, we define the delayed perturbed quantum hitting time (DPQHT) and its upper bound as the following.

Fact 7 [19] *When P is an ergodic Markov transition with positive eigenvalues, then the x-quantum hitting time for the unitary $W(P, x)$ is*

$$\text{QHT}(P, x) = \sum_{j=1}^{n-1} \frac{\nu_j^2}{\theta_j} \tag{25}$$

Proof *Since the length of the projection of $|\phi\rangle_{-u}$ to the eigenspace corresponding to α_j is ν_j^2* [19], *then by Eq. 24 we have the result as shown in Eq. 25.* □

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Proposition 2 *Given* QHT(P, x) *and* QHT(Q, x) *with* $\|P - Q\| = \|E\|$, *denote the Delayed Perturbed Quantum Hitting Time* DPQHT(P, Q, x) *that*

$$\text{DPQHT}(P, Q, x) = \text{QHT}(Q, x) - \text{QHT}(P, x).$$

By use of Fact 7, we have DPQHT(P, Q, x) bounded from above by

$$\frac{1}{\sqrt{1 - \lambda_1 - \|E\|}} - \frac{1}{\sqrt{1 - \lambda_1 + \gamma}}.$$

The eigenvalues of P_{-x} are ordered such that $1 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > 0$ and $\lambda_1 - \lambda_{n-1} = \gamma$.

Proof Based on Fact 7, we have

$$\begin{aligned}
\text{DPQHT}(P, Q, x) &= \text{QHT}(Q, x) - \text{QHT}(P, x) \\
&= \sum_{i \in \Omega} \left(\frac{\tilde{\nu}_i^2}{\tilde{\theta}_i} - \frac{\nu_i^2}{\theta_i} \right) \\
&\leq \left(\sum_{i \in \Omega} \frac{\tilde{\nu}_i^2}{\cos^{-1} \tilde{\lambda}_1} \right) - \left(\sum_{i \in \Omega} \frac{\nu_i^2}{\cos^{-1} \lambda_{n-1}} \right) \\
&\leq \left(\frac{1}{\cos^{-1}(\lambda_1 + \|E\|)} \right) - \left(\frac{1}{\cos^{-1}(\lambda_1 - \gamma)} \right) \\
&\leq \frac{1}{\sqrt{1 - \lambda_1 - \|E\|}} - \frac{1}{2\sqrt{1 - \lambda_1 + \gamma}}.
\end{aligned} \tag{26}$$

The last two inequalities are direct results from Fact 4 and the fact that $2\sqrt{1-\lambda} > \cos^{-1}\lambda > \sqrt{1-\lambda}$ for all $\lambda \in (0, 1)$. However, this bound at Eq. (26) is not tight enough.

Let us define a function $f(x) = \frac{1}{\sqrt{1-x}} - \frac{1}{\cos^{-1}(x)}$ where $0 < x < 1$. It is clear that $f(x)$ is strictly increasing since $f'(x) > 0$. Given the constraints that $1 > \lambda_1 > 0$, $\|E\| > 0$ and $\gamma \geq 0$, let $\alpha = \lambda_1 + \|E\|$ and $\beta = \lambda_1 - \gamma$. We then have

$$\left(\frac{1}{\cos^{-1}(\lambda_1 + \|E\|)} - \frac{1}{\cos^{-1}(\lambda_1 - \gamma)} - \frac{1}{\sqrt{1 - \lambda_1 - \|E\|}} + \frac{1}{\sqrt{1 - \lambda_1 + \gamma}} \right) = f(\beta) - f(\alpha) < 0$$

because $\alpha > \beta$ and the function f is strictly increasing. This guarantees that

$$\text{DPQHT}(P, Q, x) \leq \frac{1}{\sqrt{1 - \lambda_1 - \|E\|}} - \frac{1}{\sqrt{1 - \lambda_1 + \gamma}}.$$

□

As a consequence of the Proposition 2, we know

$$\text{DPQHT}(P, Q, x) \leq \frac{1}{\sqrt{1 - \lambda_1 - \|E\|}} - \frac{1}{\sqrt{1 - \lambda_1 + \gamma}} \leq \sqrt{\frac{1}{1 - \lambda_1 - \|E\|} - \frac{1}{1 - \lambda_1 + \gamma}}.$$

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This immediately implies that DPQHT(P, Q, x) is also bounded from the above by the square root of the upper bound of DPHT(P, Q, x) obtained in Proposition 1.

4 Conclusion

By quantizing a perturbed symmetric stochastic $n \times n$ matrix Q with noise E, we find an upper bound for the perturbed quantum hitting time. We also show, in Fact 5, the lower bound for the magnitude of noise when the quadratic speed-up gained from the quantum walk will be annihilated by the noise.

Furthermore we compute the upper bound for the delayed perturbed quantum hitting time based on the definition of quantum hitting time. An appropriate bound for DPQHT is

$$\frac{1}{\sqrt{1-\lambda_1-\|E\|}}-\frac{1}{\sqrt{1-\lambda_1+\gamma}}. \tag{27}$$

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