

Noiseless Subsystems for Collective Rotation Channels in Quantum Information Theory

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Abstract. Collective rotation channels are a fundamental class of channels in quantum computing and quantum information theory. The commutant of the noise operators for such a channel is a C^* -algebra which is equal to the set of fixed points for the channel. Finding the precise spatial structure of the commutant algebra for a set of noise operators associated with a channel is a core problem in quantum error prevention. We draw on methods of operator algebras, quantum mechanics and combinatorics to explicitly determine the structure of the commutant for the class of collective rotation channels.

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1. Introduction

Quantum information theory provides the underlying mathematical formalism for quantum computing and is an interesting field of research in its own right [30]. While quantum computing and communication promise far reaching applications [7, 17, 29], there are numerous technical and theoretical difficulties that must be overcome. Of particular interest is the study of quantum error correction and error prevention methods. In classical computing, the types of errors that can occur are very limited. On the other hand, the fragile nature of quantum systems shows that in quantum computing there is a much richer variety of potential errors. Fortunately, methods of quantum error correction have recently been developed showing, in principle, that these difficulties may be overcome (see [1, 16, 21, 22, 24, 33] for an introduction to the subject).

Central to quantum information theory is the analysis of *quantum channels* [30]. Mathematically, a quantum channel is given by a completely positive trace

preserving map which acts on the set of operators on a finite dimensional Hilbert space. Every channel has a family of noise operators that determine the map in a natural way. One of the most promising methods of passive quantum error correction, recently developed by the third author and others [12, 14, 18, 22, 23, 28, 41], is called the *noiseless subsystem method*. Given a quantum channel, the basic tenet of this method is to use the structure of the operator algebra defined by the commutant of the associated noise operators to prepare initial quantum states which are immune to the noise of the channel. Thus it is a fundamental problem in quantum error correction to find the structure of this ‘noise commutant’. However, let us emphasize that it is the *precise* spatial structure of this algebra that must be identified. This point is clarified in the discussion of the next section.

An important test class for the noiseless subsystem method and other quantum error correction methods is the class of *collective rotation channels* [4, 5, 13, 14, 18, 20, 23, 37, 38, 39, 40, 42]. This class has its roots in the depths of quantum mechanics, specifically in the study of angular momentum at the atomic level (see for example [10]). A realistic physical situation where these channels arise occurs when quantum information, encoded as light pulses, is transmitted through an optical fibre [14, 36]. In such a situation, the fibre can produce a ‘collective rotation’ of the information.

In [8, 27] it was shown that when a channel is unital, which is the case for collective rotation channels, the noise commutant is a finite dimensional C^* -algebra which is equal to the fixed point set for the channel. Based on operator algebra techniques, the paper [18] derives an algorithm for computing the commutant structure in the most general setting. However, for particular cases such as the channels considered here, the required computations can become unwieldy.

In this paper, based on the theory of operator algebras and quantum mechanics, we compute the noise commutant structure for the class of collective rotation channels. We provide a constructive proof which yields a simple visual interpretation based on Pascal’s triangle. This result may also be derived from well-known representation theory techniques; however, our direct operator theory cum quantum mechanical approach is novel and offers a new perspective on the general problem.

The next section contains a brief review of the material we require from the theories of operator algebras and quantum information. In the third section we define the collective rotation channels and establish some basic properties. The fourth section contains the commutant structure theorem for the ‘qubit’ case (Theorem 4.1). Finally, we conclude the paper by presenting a commutant structure theorem for more general classes of collective rotation channels (Theorem 5.3).

One final comment. A study of the quantum information and quantum computing literature reveals that many techniques from operator theory and operator algebras have been, or could be, used to build mathematical foundations for the physical theories in these areas. An idea we wish to promote is that there is a wealth of interesting mathematics to be found in this young field.

2. Background

Motivated by the postulates of quantum mechanics, an assumption typically made in quantum information theory is that every quantum operation on a closed quantum system is reversible [10, 30]. Mathematically, this statement means that the operation is described by unitary evolution; in other words, there is a unitary operator U on a Hilbert space \mathcal{H} such that the operation is implemented by the conjugation map $\rho \mapsto U\rho U^\dagger$ where ρ is an operator on \mathcal{H} . (Here we use the physics convention U^\dagger for conjugate transpose.) Often ρ is a density operator, a positive operator with trace equal to one, that corresponds to the initial state of the quantum system of interest, but in our analysis there is no loss of generality in considering evolution of any operator under the quantum operation. Further note that U can be restricted to the special unitary group $SU(N)$, where $N = \dim(\mathcal{H})$, since the evolution $\rho \mapsto U\rho U^\dagger$ is unaffected by the multiplication of U by a complex phase.

Of course, in practice a given quantum operation will not be reversible because of interactions with the environment. In this more realistic setting the quantum operation is regarded as acting on a closed quantum system that contains the original as a subsystem. The mathematical formalism for this is given by completely positive maps [9, 25, 31, 32] and the Stinespring dilation theorem [34]. Specifically, every quantum operation is represented mathematically by a quantum channel.

Given a (finite dimensional) Hilbert space \mathcal{H} , a *quantum channel* is a map \mathcal{E} which acts on the set $\mathcal{B}(\mathcal{H})$ of all operators on \mathcal{H} and is completely positive and trace preserving. For each channel \mathcal{E} there is a set of (non-unique) *noise operators* [9, 25] $\{A_1, \dots, A_n\}$ that determine the map through the equation

$$\mathcal{E}(\rho) = \sum_{k=1}^n A_k \rho A_k^\dagger \quad \text{for } \rho \in \mathcal{B}(\mathcal{H}). \quad (1)$$

Physically, the associated quantum operation can be regarded as determined by a compression of the Stinespring unitary dilation, that acts on a larger closed quantum system, of the completely positive map (1). Trace preservation is equivalent to the noise operators satisfying the equation

$$\sum_{k=1}^n A_k^\dagger A_k = \mathbb{1},$$

where $\mathbb{1}$ is the identity operator on \mathcal{H} . The channel is unital if also,

$$\mathcal{E}(\mathbb{1}) = \sum_{k=1}^n A_k A_k^\dagger = \mathbb{1}.$$

Let $\text{Fix}(\mathcal{E}) = \{\rho \in \mathcal{B}(\mathcal{H}) : \mathcal{E}(\rho) = \rho\}$ be the fixed point set for \mathcal{E} and let \mathcal{A} be the algebra generated by A_1, \dots, A_n from (1). This is called the *interaction algebra* in quantum information theory [23]. In general, $\text{Fix}(\mathcal{E})$ is just a \dagger -closed subspace of $\mathcal{B}(\mathcal{H})$, but it was shown (independently) in [8] and [27] that, in the

case of a unital channel \mathcal{E} , the so-called *noise commutant* $\mathcal{A}' = \{\rho \in \mathcal{B}(\mathcal{H}) : \rho A_k = A_k \rho, k = 1, \dots, n\}$ coincides with this set:

$$\text{Fix}(\mathcal{E}) = \mathcal{A}'.$$

In particular, $\text{Fix}(\mathcal{E}) = \mathcal{A}'$ is a \dagger -closed operator algebra (a finite dimensional C^* -algebra [3, 11]). Further, the von Neumann double commutant theorem shows how the algebra $\mathcal{A} = \mathcal{A}'' = \text{Fix}(\mathcal{E})'$ only depends on the channel; that is, it is independent of the choice of noise operators that determine the channel as in (1).

It is a fundamental result in finite dimensional C^* -algebra theory [3, 11, 35] that every such algebra is unitarily equivalent to an orthogonal direct sum of ‘ampliated’ full matrix algebras; i.e., there is a unitary operator U such that

$$U\mathcal{A}U^\dagger = \sum_{k=1}^d \oplus (\mathbb{1}_{m_k} \otimes \mathcal{M}_{n_k}),$$

where \mathcal{M}_{n_k} is the full matrix operator algebra $\mathcal{B}(\mathbb{C}^{n_k})$. The numbers m_k in this decomposition correspond to the multiplicities in the C^* -algebra representation that gives \mathcal{A} . With this form for \mathcal{A} given, the structure of the commutant up to unitary equivalence is easily computed by

$$\mathcal{A}' \cong \sum_{k=1}^d \oplus (\mathcal{M}_{m_k} \otimes \mathbb{1}_{n_k}). \quad (2)$$

(See [14, 18, 20, 37, 39, 40] for more detailed discussions in connection with quantum information theory.) The projections onto the $m_k n_k$ dimensional subspaces associated with the summands in this decomposition are the minimal central projections for \mathcal{A} (and \mathcal{A}'). Each such projection is the sum of n_k rank m_k projections which are minimal reducing for \mathcal{A}' . We describe this situation by saying these n_k projections are ‘linked’ inside \mathcal{A}' .

On the other hand, for a given quantum channel \mathcal{E} with noise operators $\{A_k\}$, the noise commutant \mathcal{A}' plays a significant role in quantum error prevention. The structure of this commutant can be used to prepare density operators, which encode the state of a given quantum system, for use in the noiseless subsystem method of error correction. This is a passive method of quantum error correction, in the sense that such operators will remain immune to the effects of the noise operators, or ‘errors’ of the channel, without active intervention. But more is true. The algebra structure discussed above shows that quantum operations may be performed on such a subsystem, provided the corresponding unitary operators belong to the commutant. Keeping in mind our earlier description of an optical fibre, the reader can imagine a situation where it is desirable to transfer quantum information through the fibre such that the information remains immune to the errors of collective rotations produced by the fibre.

As discussed above, understanding the structure of \mathcal{A}' is of fundamental importance in quantum error correction. But there is an operator algebra subtlety here which is worth emphasizing. Typically, it is not feasible in this setting to wash

away the particular representation which gives \mathcal{A}' with $*$ -isomorphisms, unitary equivalences, etc., as is the custom in operator algebra theory. Indeed, by the very nature of the problems, it is the precise spatial algebra structure of \mathcal{A}' which must be identified, ampliations included.

The basic problem of computing \mathcal{A}' was addressed in [18] for the general case of a unital quantum channel. We also mention more recent work [42] where computer algorithms have been written for this and other related purposes. However, in particular cases, such as the class of channels considered in this paper, a more delicate approach based on special properties of the class can be exploited to find this structure more directly and efficiently.

3. Collective Rotation Channels

Let $\{|\frac{-1}{2}\rangle, |\frac{1}{2}\rangle\}$ be a fixed orthonormal basis for 2-dimensional Hilbert space $\mathcal{H}_2 = \mathbb{C}^2$, corresponding to the classical base states in a two level quantum system (e.g. the ground and excited states of an electron in a Hydrogen atom). Note that such a basis is usually written as $\{|0\rangle, |1\rangle\}$, but the $\frac{-1}{2}, \frac{1}{2}$ notation is more convenient for the combinatorics below. A ‘qubit’ or ‘quantum bit’ of information is given by a unit vector $|\psi\rangle = \alpha|\frac{-1}{2}\rangle + \beta|\frac{1}{2}\rangle$ inside \mathcal{H}_2 . When both α and β are non-zero, $|\psi\rangle$ is said to be a *superposition* of $|\frac{-1}{2}\rangle$ and $|\frac{1}{2}\rangle$.

We shall make use of the abbreviated form from quantum mechanics for the associated standard orthonormal basis for $\mathcal{H}_{2^n} = (\mathbb{C}^2)^{\otimes n} \simeq \mathbb{C}^{2^n}$. For instance, the basis for \mathcal{H}_4 is given by

$$\left\{ |ij\rangle : i, j \in \left\{ \frac{-1}{2}, \frac{1}{2} \right\} \right\}$$

where $|ij\rangle$ is the vector tensor product $|ij\rangle \equiv |i\rangle|j\rangle \equiv |i\rangle \otimes |j\rangle$.

Let $\{\sigma_x, \sigma_y, \sigma_z\}$ be the spin-1/2 Pauli matrices given by

$$\sigma_x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Further let $\mathbb{1}_2$ be the 2×2 identity matrix. We shall regard these as the matrix representations for operators acting on \mathcal{H}_2 with respect to $\{|\frac{-1}{2}\rangle, |\frac{1}{2}\rangle\}$. The Pauli matrices satisfy the following commutation relations:

- (1) $[\sigma_x, \sigma_y] = i\sigma_z$
- (2) $[\sigma_z, \sigma_x] = i\sigma_y$
- (3) $[\sigma_y, \sigma_z] = i\sigma_x$.

These are the canonical commutation relations which define the Lie algebra $su(2)$, given by the linear space $r_x\sigma_x + r_y\sigma_y + r_z\sigma_z = \vec{r} \cdot \vec{\sigma}$ with $(r_x, r_y, r_z) \in \mathbb{R}$. This algebra is the generator of the Lie group $SU(2)$ as the manifold of 2×2 unitary matrices with unit determinant and is isomorphic to the manifold $\{\exp(-i2\pi\vec{r} \cdot \vec{\sigma}) : \|\vec{r}\| \leq 1\}$. The group $SU(2)$ is referred to as the rotation group as it is homeomorphic to $O(3)$, the rotational group in three-dimensional space. Note that

a rotation is the most general transformation which can be performed on a closed two-dimensional quantum system.

Now let $n \geq 1$ be a fixed positive integer. Define operators $\{J_z^{(k)} : 1 \leq k \leq n\}$ on \mathcal{H}_{2^n} by

$$J_z^{(1)} = \sigma_z \otimes (\mathbb{1}_2)^{\otimes(n-1)}, \quad J_z^{(2)} = \mathbb{1}_2 \otimes \sigma_z \otimes (\mathbb{1}_2)^{\otimes(n-2)}, \quad \dots,$$

where we use the standard ordering $(a_{kl}B)_{kl}$ for the tensor product of matrices $A \otimes B$. Similarly define $\{J_x^{(k)}, J_y^{(k)} : 1 \leq k \leq n\}$. Then the *collective rotation generators* $\{J_x, J_y, J_z\}$ are given by

$$J_x = \sum_{k=1}^n J_x^{(k)}, \quad J_y = \sum_{k=1}^n J_y^{(k)}, \quad J_z = \sum_{k=1}^n J_z^{(k)}.$$

Let us set down the fundamental commutation relations satisfied by these operators [10].

Proposition 3.1. *The following relations hold for $\{J_x, J_y, J_z\}$:*

$$\begin{aligned} (1)' \quad [J_x, J_y] &= iJ_z \\ (2)' \quad [J_z, J_x] &= iJ_y \\ (3)' \quad [J_y, J_z] &= iJ_x. \end{aligned}$$

Proof. These identities easily follow from corresponding equations for $J_x^{(k)}, J_y^{(k)}, J_z^{(k)}$, with $1 \leq k \leq n$, which are simple consequences of the commutation relations (1), (2), (3) satisfied by the spin-1/2 Pauli matrices. \square

Note 3.2. Observe that Proposition 3.1 shows $\{J_x, J_y, J_z\}$ determine a 2^n -dimensional representation of $su(2)$.

In what follows, much of the analysis will be focused on the operators

$$J_+ = J_x + iJ_y \quad \text{and} \quad J_- = J_x - iJ_y = J_+^\dagger.$$

We shall also consider the so-called *J-total* operator J^2 defined by

$$J^2 = J_x^2 + J_y^2 + J_z^2.$$

The J^2 notation comes from the fact that this operator is conventionally defined as a vector product of matrices [10].

Intuitively, the collective rotation channel is one where every qubit undergoes the same unknown rotation. Let us formalize this notion. Consider a channel $\mathcal{E}_{U^{\otimes n}} : \mathcal{B}(\mathcal{H}_{2^n}) \rightarrow \mathcal{B}(\mathcal{H}_{2^n})$ defined as $\mathcal{E}_{U^{\otimes n}}(T) = (U^{\otimes n})T(U^{\otimes n})^\dagger$ for $U \in SU(2)$. This is a collective rotation of n qubits which can also be written $\mathcal{E}_{U^{\otimes n}}(T) = \exp(-i2\pi\vec{r} \cdot \vec{J})T \exp(i2\pi\vec{r} \cdot \vec{J})$, where $\vec{r} \cdot \vec{J} = r_x J_x + r_y J_y + r_z J_z$ and $U = \exp(-i2\pi\vec{r} \cdot \vec{\sigma})$. Hence the appellation *collective rotation generators*.

But here, the specific rotation U is unknown and chosen at random over $SU(2)$ according to a given probability density $P(\vec{r})$, for instance the distribution

corresponding to Haar measure on $SU(2)$. Then, the n -qubit collective rotation channel (relative to $P(\vec{r})$) can be written as

$$\mathcal{E}_n(T) = \int_{\{|\vec{r}|\leq 1\}} \exp(-i2\pi\vec{r}\cdot\vec{J}) T \exp(i2\pi\vec{r}\cdot\vec{J}) P(\vec{r}) d\vec{r}; \tag{3}$$

it is a weighted average of all collective rotations. By the symmetry of the integrated region, it can be shown that this unital channel can also be expressed in a more conventional form,

$$\mathcal{E}_n(T) = E_x T E_x^\dagger + E_y T E_y^\dagger + E_z T E_z^\dagger, \tag{4}$$

where the noise operators are defined as

$$E_x = \frac{1}{\sqrt{3}} \exp(i\theta_x J_x), \quad E_y = \frac{1}{\sqrt{3}} \exp(i\theta_y J_y), \quad E_z = \frac{1}{\sqrt{3}} \exp(i\theta_z J_z),$$

and $\theta_k, k = x, y, z$, are angles determined by the probability distribution $P(\vec{r})$ [10].

It is not hard to see that our analysis is independent of the particular choices for these angles, provided each θ_k is non-zero. Indeed, through a standard functional calculus argument from operator algebra, it can be seen that the interaction algebras generated by the J_k and E_k coincide, whatever the choice of θ_k ;

$$\mathcal{A}_n \equiv \text{Alg}\{J_x, J_y, J_z\} = \text{Alg}\{J_+, J_-, J_z\} = \text{Alg}\{E_x, E_y, E_z\}. \tag{5}$$

In particular, as observed in [18], the fixed point set of this channel is determined by the original rotation generators.

Proposition 3.3. *Let $n \geq 1$ be a positive integer. Then*

$$\text{Fix}(\mathcal{E}_n) = \mathcal{A}'_n = \{E_x, E_y, E_z\}' = \{J_x, J_y, J_z\}'.$$

Further, this commutant may be computed by considering the joint commutant of any pair from $\{J_x, J_y, J_z\}$.

4. Commutant Structure Theorem

Given a positive integer $n \geq 1$, let Δ_n denote the graph of $\binom{n}{\cdot}$; that is, the graph of the n th line in Pascal's triangle. (See the example below for a pictorial perspective.) Let

$$\mathcal{J}_n = \begin{cases} \{0, 1, \dots, \frac{n}{2}\} & \text{if } n \text{ is even} \\ \{\frac{1}{2}, \frac{3}{2}, \dots, \frac{n}{2}\} & \text{if } n \text{ is odd} \end{cases}$$

Observe that the cardinality of \mathcal{J}_n is equal to the number of steps up one side of Δ_n .

Theorem 4.1. *Let \mathcal{E}_n be the collective rotation channel for a fixed positive integer $n \geq 1$. Then*

$$\text{Fix}(\mathcal{E}_n) = \mathcal{A}'_n = \sum_{j \in \mathcal{J}_n} \oplus \mathcal{A}'_{(j)}, \tag{6}$$

where $\mathcal{A}'_{(j)}$ is a \mathbb{C}^* -subalgebra of \mathcal{A}'_n given, up to unitary equivalence, by

$$\mathcal{A}'_{(j)} \simeq \mathcal{M}_{p_j} \otimes \mathbb{1}_{q_j} \quad \text{for } j \in \mathcal{J}_n,$$

with $p_{\frac{n}{2}} = 1$ and for $j \in \mathcal{J}_n$, $j < \frac{n}{2}$,

$$p_j = \binom{n}{j + \frac{n}{2}} - \binom{n}{j + \frac{n}{2} + 1} = \binom{n+1}{j + \frac{n}{2} + 1} \frac{q_j}{n+1},$$

where

$$q_j = 2j + 1 \quad \text{for } j \in \mathcal{J}_n.$$

In the proof below we shall explicitly identify the spatial decomposition that yields this decomposition of \mathcal{A}'_n . Recall that this is necessary for using the noiseless subsystem approach to quantum error correction. Before proving this theorem, let us illustrate how Δ_n gives a visual method for determining the commutant structure. For the sake of brevity, let us focus on a single case, the $n = 4$ collective rotation channel \mathcal{E}_4 .

Example 4.2. In the $n = 4$ case we have $\mathcal{J}_4 = \{0, 1, 2\}$ and $p_0 = 2, p_1 = 3, p_2 = 1$ and $q_0 = 1, q_1 = 3, q_2 = 5$. The theorem states that

$$\text{Fix}(\mathcal{E}_4) = \mathcal{A}'_4 = \mathcal{A}'_{(0)} \oplus \mathcal{A}'_{(1)} \oplus \mathcal{A}'_{(2)},$$

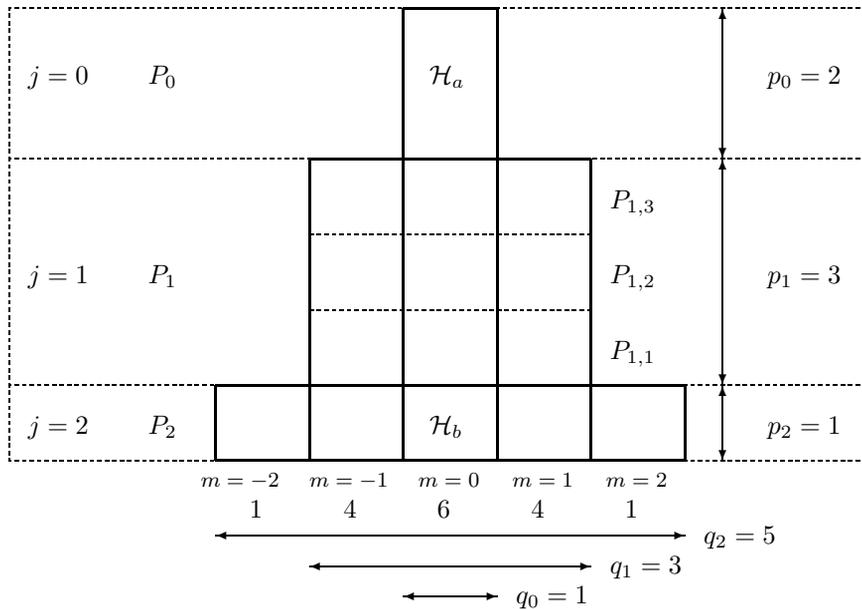
with each $\mathcal{A}^{(j)}$ a subalgebra of \mathcal{A}'_4 unitarily equivalent to

$$\begin{aligned} \mathcal{A}'_{(0)} &\simeq \mathbb{C} \otimes \mathbb{1}_5 \simeq \mathbb{C}\mathbb{1}_5 \\ \mathcal{A}'_{(1)} &\simeq \mathcal{M}_3 \otimes \mathbb{1}_3 \\ \mathcal{A}'_{(2)} &\simeq \mathcal{M}_2 \otimes \mathbb{1}_1 \simeq \mathcal{M}_2. \end{aligned}$$

Consider the structure of Δ_4 as in Figure 1. The number p_j corresponds to the ‘height’ of the j th horizontal bar (counting top-down), and q_j equals the number of blocks inside this bar. Spatially, the vertical bars correspond to the eigenspaces for J_z for the eigenvalues $m = -2, -1, 0, 1, 2$ (with eigenspace projections Q_m in the proof below), which have respective multiplicities 1, 4, 6, 4, 1. The horizontal bars correspond to eigenspaces of J^2 (Corollary 4.10). The corresponding eigenspace projections P_0, P_1, P_2 are the minimal central projections for \mathcal{A}_4 and \mathcal{A}'_4 .

To see how the blocks correspond to subspaces, the subspace \mathcal{H}_a , as an example, for the top box in Δ_4 is the joint eigenspace for J_z and J^2 , corresponding to $m = 0$ and $j = 0$ with our notation below. Each of the j th horizontal bars further breaks up into smaller horizontal bars, for instance $P_1 = \sum_{k=1}^3 P_{1,k}$. The subspaces $\{P_{j,k}\mathcal{H}\}$ form the maximal family of minimal reducing subspaces for \mathcal{A}_4 as outlined below. On the other hand, the corresponding family for \mathcal{A}'_4 is given by the vertical blocks inside the j th horizontal bar. For example, the projection onto \mathcal{H}_b and the projections onto its other four counterparts in the $j = 2$ bar (which are all 1-dimensional because they lie in the $j = 2$ bar) are the family of minimal \mathcal{A}'_4 -reducing subspaces supported on P_2 .

FIGURE 1



We now turn to the proof of Theorem 4.1. Let $n \geq 1$ be a fixed positive integer. We shall find the structure of \mathcal{A}'_n by first computing the structure of \mathcal{A}_n . We begin by showing how the numeric distribution of the eigenvalues for J_z is linked with Δ_n . In what follows, we use the abbreviated Dirac notation to denote the standard orthonormal basis for $\mathcal{H} \equiv \mathcal{H}_{2^n} = \mathbb{C}^{2^n}$ with $|\frac{-1}{2}\rangle, |\frac{1}{2}\rangle$ corresponding to the base states of the two-level quantum system ($d = 2$ with our notation in the next section);

$$\left\{ |\vec{i}\rangle = |i_1 i_2 \cdots i_n\rangle : i_j \in \left\{ \frac{-1}{2}, \frac{1}{2} \right\}, 1 \leq j \leq n \right\}.$$

Lemma 4.3. For $m = -\frac{n}{2}, -\frac{n}{2} + 1, \dots, \frac{n}{2}$ consider the subspaces of \mathcal{H} given by

$$\mathcal{V}_m = \text{span} \{ |\vec{i}\rangle : |\vec{i}| = m \},$$

where $|\vec{i}| = \sum_{j=1}^n i_j$. Then $\mathcal{H} = \sum_{m=-\frac{n}{2}}^{\frac{n}{2}} \oplus \mathcal{V}_m$ and

$$\dim \mathcal{V}_m = \binom{n}{m + \frac{n}{2}} \text{ for } -\frac{n}{2} \leq m \leq \frac{n}{2}.$$

Further, \mathcal{V}_m is an eigenspace for J_z corresponding to the eigenvalue

$$\lambda = m \text{ for } -\frac{n}{2} \leq m \leq \frac{n}{2}.$$

Proof. The spatial decomposition of \mathcal{H} is easy to see and the dimensions of the \mathcal{V}_m follow from simple combinatorics. For the eigenvalue connection with J_z , observe that for $|\vec{i}\rangle = m$ we have

$$J_z|\vec{i}\rangle = \sum_{k=1}^n J_z^{(k)}|\vec{i}\rangle = \sum_{k=1}^n i_k|\vec{i}\rangle = |\vec{i}\rangle|\vec{i}\rangle = m|\vec{i}\rangle. \quad \square$$

For $-\frac{n}{2} \leq m \leq \frac{n}{2}$, let Q_m be the orthogonal projection of \mathcal{H} onto $\mathcal{V}_m \equiv Q_m\mathcal{H}$.

Lemma 4.4. *Given $-\frac{n}{2} \leq m \leq \frac{n}{2}$, we have*

$$J_+Q_m = \begin{cases} Q_{m+1}J_+Q_m & \text{if } m < \frac{n}{2} \\ 0 & \text{if } m = \frac{n}{2} \end{cases}$$

and

$$J_-Q_m = \begin{cases} Q_{m-1}J_-Q_m & \text{if } m > -\frac{n}{2} \\ 0 & \text{if } m = -\frac{n}{2} \end{cases}$$

Proof. Let $|\psi\rangle$ belong to $Q_m\mathcal{H}$. Then $J_z|\psi\rangle = m|\psi\rangle$. But notice that

$$\begin{aligned} J_zJ_+ &= J_z(J_x + iJ_y) = J_xJ_z + iJ_y + iJ_yJ_z + J_x \\ &= J_+(J_z + \mathbb{1}). \end{aligned}$$

Thus $J_zJ_+|\psi\rangle = (m+1)J_+|\psi\rangle$ when $m < \frac{n}{2}$, so that $J_+|\psi\rangle$ belongs to $Q_{m+1}\mathcal{H}$. The corresponding identities for J_- are proved in a similar fashion and for convenience the identities $J_+Q_m = 0 = J_-Q_{-m}$, $m = \frac{n}{2}$, will be observed in the discussion which follows. \square

Next we shall derive a spatial decomposition of \mathcal{H} which will allow us to connect with the structure of Δ_n . Let

$$|0_L\rangle \equiv \left| \frac{n}{2}, -\frac{n}{2}, 1 \right\rangle$$

be a (unit) eigenvector for J_z for the eigenvalue $m = -\frac{n}{2}$. The span of $|0_L\rangle$ will be identified with the ‘bottom left corner’ of Δ_n , see Corollary 4.6 below. To simplify notation, let $ns = \frac{n}{2}$ (the use of this notation will become clear in the next section). Lemma 4.4 shows that $J_+|0_L\rangle \equiv |ns, -ns + 1, 1\rangle$ is an eigenvector of J_z for the eigenvalue $m = -ns + 1$. Similarly, the vectors

$$J_+^p|0_L\rangle \equiv |ns, -ns + p, 1\rangle \quad \text{for } 0 \leq p < q_{ns},$$

are non-zero and belong to \mathcal{V}_{-ns+p} .

Let $\{|ns - 1, -ns + 1, \mu\rangle\}_\mu$ be an orthonormal basis for

$$\mathcal{V}_{-ns+1} \ominus \text{span}\{|ns, -ns + 1, 1\rangle\}.$$

At this stage the choice of this basis, indexed by μ , is arbitrary. Now inductively, if we are given $j \in \mathcal{J}_n$ with $j < ns$, let $\{|j, m = -j, \mu\rangle\}_\mu$ be an (arbitrary) orthonormal basis for

$$\mathcal{V}_{-j} \ominus \text{span}\{|j', -j, \mu\rangle : j < j' \leq ns\},$$

where $|j', -j, \mu\rangle = J_+^{(j'-j)}|j', -j', \mu\rangle$.

Notice that

$$J_- |j, m = -j, \mu\rangle = 0 \quad \text{for all } j, \mu. \tag{7}$$

Indeed, by choice of the vectors $|j, -j, \mu\rangle$ and from the ‘eigenspace shifting’ of Lemma 4.4, it follows that each $|j, -j, \mu\rangle$ is orthogonal to the range space of J_+ . Thus J_- annihilates the left hand steps of Δ_n , which is the content of (7). From this we also have

$$J_+ |j, m = j, \mu\rangle = J_+^{2j} |j, m = -j, \mu\rangle = 0 \quad \text{for all } \mu. \tag{8}$$

In other words, from the Δ_n picture given by Corollary 4.6 below, J_+ annihilates the right hand side blocks of Δ_n .

Thus, in summary we have a collection of vectors $|j, m, \mu\rangle$ (which turn out to form an orthogonal basis for \mathcal{H}) such that: j belongs to \mathcal{J}_n , for fixed j the range of m is $-j \leq m \leq j$, for each j, m, μ ,

$$|j, m, \mu\rangle = J_+^{(m+j)} |j, -j, \mu\rangle = J_-^{(m-j)} |j, j, \mu\rangle,$$

and for a given m, j pair the index μ has p_j possible values.

For fixed j, μ let $\mathcal{H}(j, \mu)$ be the subspace defined by

$$\mathcal{H}(j, \mu) = \text{span} \{ |j, m, \mu\rangle : -j \leq m \leq j \}.$$

Such a subspace corresponds to a horizontal slice of the ‘ j th horizontal bar’ in Δ_n . From Corollary 4.6, it follows that these subspaces are pairwise orthogonal for distinct pairs j, μ . (This justifies the use of the orthogonal sum symbol \oplus in the following statement.)

Lemma 4.5. *The operator J^2 belongs to the centre of \mathcal{A} ; that is,*

$$J^2 \in \mathcal{A}_n \cap \mathcal{A}'_n.$$

Consider the subspaces

$$\mathcal{W}_j = \sum_{\mu} \oplus \mathcal{H}(j, \mu) \quad \text{for } j \in \mathcal{J}_n.$$

Then the restriction of J^2 to each of these subspaces is a constant operator; i.e., there are scalars λ_j such that

$$J^2|_{\mathcal{W}_j} = \lambda_j \mathbb{1}_{\mathcal{W}_j} \quad \text{for } j \in \mathcal{J}_n.$$

Further, these scalars satisfy $\lambda_{j_1} \neq \lambda_{j_2}$ for $j_1 \neq j_2$.

Proof. By definition J^2 belongs to \mathcal{A} . We show that J^2 commutes with J_x . The J_y and J_z cases are similar. Observe that

$$\begin{aligned} [J_x, J^2] &= [J_x, J_y^2 + J_z^2] = [J_x, J_y^2] + [J_x, J_z^2] \\ &= J_y [J_x, J_y] + [J_x, J_y] J_y + J_z [J_x, J_z] + [J_x, J_z] J_z \\ &= J_y (iJ_z) + (iJ_z) J_y + J_z (-iJ_y) + (-iJ_y) J_z = 0. \end{aligned}$$

Consider a vector $|j, -j, \mu\rangle$ in the left most block of \mathcal{W}_j . Observe that $J^2 = J_+J_- + J_z^2 - J_z$, and hence

$$J^2|j, -j, \mu\rangle = (J_z^2 - J_z)|j, -j, \mu\rangle = (j^2 + j)|j, -j, \mu\rangle.$$

As J^2 belongs to \mathcal{A}'_n , we have $J^2J_+ = J_+J^2$. Thus, given a typical basis vector $J_+^{(m+j)}|j, -j, \mu\rangle = |j, m, \mu\rangle$ inside \mathcal{W}_j compute

$$\begin{aligned} J^2(J_+^{(m+j)}|j, -j, \mu\rangle) &= J_+^{(m+j)}J^2|j, -j, \mu\rangle \\ &= (j^2 + j)J_+^{(m+j)}|j, -j, \mu\rangle = (j^2 + j)|j, m, \mu\rangle. \end{aligned}$$

It follows that the corresponding restrictions of J^2 satisfy $J^2|_{\mathcal{W}_j} = (j^2 + j)\mathbb{1}_{\mathcal{W}_j}$, and the scalars $\lambda_j = j^2 + j$ are different for distinct values of j . \square

Corollary 4.6. *The vectors $\{|j, m, \mu\rangle\}_{j,m,\mu}$ are non-zero and form an orthogonal basis for \mathcal{H} . Thus,*

$$\mathcal{H} = \sum_j \oplus \mathcal{W}_j = \sum_{j,\mu} \oplus \mathcal{H}(j, \mu),$$

and the subspaces $\{\mathcal{W}_j\}$ are the eigenspaces for J^2 .

Proof. These vectors are clearly all non-zero by the above discussions. Consider two vectors from this set, $J_+^{p_i}|j_i, m_i, \mu_i\rangle$ for $i = 1, 2$. Then

$$\langle j_1, m_1, k_1 | J_+^{p_2 - p_1} | j_2, m_2, k_2 \rangle = 0 \quad \text{if } (j_1, m_1, k_1) \neq (j_2, m_2, k_2).$$

This follows from the choice of the vectors $|j, m, \mu\rangle$, the relations

$$J_+J_- = J_x^2 + J_y^2 - J_z = J^2 - J_z^2 - J_z, \quad (9)$$

$$J_-J_+ = J_x^2 + J_y^2 + J_z = J^2 - J_z^2 + J_z, \quad (10)$$

and the connections with the eigenspaces for J_z, J^2 given by Lemma 4.3 and Lemma 4.5. \square

The following perspective on the actions of J_+ and J_- will be useful below.

Lemma 4.7. *For all j, μ , the operators J_+ and $J_- = J_+^\dagger$ act as weighted shifts on the standard basis for $\mathcal{H}(j, \mu)$.*

Proof. Recall that $J_-|j, m = -j, \mu\rangle = 0$ since $|j, -j, \mu\rangle$ belongs to the orthocomplement of the range of $J_-^\dagger = J_+$; that is, $\langle j, -j, \mu | J_+ \psi \rangle = 0$ for all $|\psi\rangle \in \mathcal{H}$. Thus, by equation (10) and Lemma 4.3 and Lemma 4.5, for $p \geq 1$ there is a scalar c with

$$\begin{aligned} J_- J_+^{(m+j)} |j, m, \mu\rangle &= (J^2 - J_z^2 - J_z) J_+^{(m+j-1)} |j, m, \mu\rangle \\ &= c J_+^{p-1} |j, m-1, \mu\rangle. \end{aligned}$$

In particular, J_- acts as a backward shift on the (orthogonal) basis $\{|j, m, \mu\rangle : -j \leq m \leq j\}$ for $\mathcal{H}(j, \mu)$ with $J_-|j, -j, \mu\rangle = 0$. Similarly, by using (9) it can be seen that J_+ acts as the forward shift on this basis with $J_+|j, j, \mu\rangle = 0$.

Hence, when the basis $\{|j, m, \mu\rangle : -j \leq m \leq j\}$ is normalized to turn it into an orthonormal basis for $\mathcal{H}(j, \mu)$, we see that J_+ (respectively J_-) acts as a forward (respectively backward) weighted shift on this basis. \square

The following result shows that the family of mutually orthogonal subspaces $\mathcal{H}(j, \mu)$ forms the (unique) maximal family of minimal reducing subspaces for \mathcal{A}_n which determine the minimal central projections.

Lemma 4.8. *For all j, μ , the subspace $\mathcal{H}(j, \mu)$ is a minimal \mathcal{A}_n -reducing subspace.*

Proof. First note that $\mathcal{H}(j, \mu)$ is clearly reducing for J_z (i.e. invariant for both J_z and J_z^\dagger). Also, Lemma 4.7 shows that $\mathcal{H}(j, \mu)$ reduces J_+ and J_- . Hence $\mathcal{H}(j, \mu)$ is a reducing subspace for $\mathcal{A}_n = \text{Alg}\{J_+, J_-, J_z\}$.

To see minimality, fix j, μ and let $|\psi\rangle$ be a non-zero vector inside $\mathcal{H}(j, \mu)$. Then by Lemma 4.7 there is a $p \geq 0$ such that $J_-^p|\psi\rangle$ is a non-zero multiple of $|j, -j, \mu\rangle$. Hence, each basis vector $|j, m, \mu\rangle$, for $-j \leq m \leq j$, belongs to the subspace $\mathcal{A}_n|\psi\rangle = \mathcal{H}(j, \mu)$, and it follows that $\mathcal{H}(j, \mu)$ is minimal \mathcal{A}_n -reducing. \square

The structure of Δ_n determines which of the $\mathcal{H}(j, \mu)$ sum to give the family of minimal central projections. Recall that the minimal central projections of \mathcal{A}_n and \mathcal{A}'_n are the same since $\mathcal{A}_n \cap \mathcal{A}'_n = (\mathcal{A}'_n)' \cap \mathcal{A}'_n$.

Lemma 4.9. *For each j, μ let $P_{j,\mu}$ be the projection of \mathcal{H} onto $\mathcal{H}(j, \mu) \equiv P_{j,\mu}\mathcal{H}$. Then the minimal central projections for \mathcal{A}_n and \mathcal{A}'_n are $\{P_j\}$ where*

$$P_j = \sum_{\mu} P_{j,\mu},$$

and hence $\mathcal{W}_j = P_j\mathcal{H} = \sum_{\mu} P_{j,\mu}\mathcal{H}$.

Proof. The projections $P_{j,\mu}$ form the (unique) maximal family of non-zero minimal reducing projections for \mathcal{A}_n . Thus, the minimal central projections for \mathcal{A}_n are given by sums of the $P_{j,\mu}$, and so we must find which subsets of the $P_{j,\mu}$ are linked inside \mathcal{A}_n (see (2)). Since linked projections amongst the $P_{j,\mu}$ necessarily have the same rank, it is enough to fix j and consider the family $\{P_{j,\mu}\}_{\mu}$.

In fact, we claim that the entire family $\{P_{j,\mu}\}_{\mu}$ is linked inside \mathcal{A}_n . To see this, it is sufficient, and best for use in the noiseless subsystem method, to exhibit bases for $P_{j,\mu}\mathcal{H}$ which allow us to view the links explicitly. By construction, the basis $\{|j, m, \mu\rangle : -j \leq m \leq j\}_{\mu}$ for $P_{j,\mu}\mathcal{H}$ used in the analysis above is such a basis. Indeed, we may compute that

$$\langle j, m, \mu_1 | J_-^{p_1} A J_+^{p_2} | j, m, \mu_1 \rangle = \langle j, m, \mu_2 | J_-^{p_1} A J_+^{p_2} | j, m, \mu_2 \rangle, \tag{11}$$

for all possible choices of μ_1, μ_2, p_1, p_2 and $A \in \mathcal{A}_n$. Recall that \mathcal{A}_n is generated by J_+, J_-, J_z as an algebra. By design, (11) is evident for A equal to one of these generators, for any monomial in them, and hence, when extending by linearity, for any element of \mathcal{A}_n . It follows that for all j , the projection $P_j = \sum_{\mu} P_{j,\mu}$ is a minimal central projection for \mathcal{A}_n (and \mathcal{A}'_n). \square

Proof of Theorem 4.1. By the previous result \mathcal{A}_n has a block diagonal decomposition $\mathcal{A}_n = \sum_{j \in \mathcal{J}_n} \oplus \mathcal{A}_{(j)}$, where each $\mathcal{A}_{(j)}$ is a *subalgebra* of \mathcal{A}_n which is unitarily equivalent to $\mathcal{A}_{(j)} \simeq \mathbb{1}_{p_j} \otimes \mathcal{M}_{q_j}$, since $\text{rank } P_{j,\mu} = q_j = \dim P_{j,\mu} \mathcal{H}$ for all μ and there are p_j linked projections $\{P_{j,\mu}\}_\mu$. Therefore, the commutant $\text{Fix}(\mathcal{E}_n) = \mathcal{A}'_n$ may be obtained by

$$\text{Fix}(\mathcal{E}_n) = \mathcal{A}'_n = \sum_{j \in \mathcal{J}_n} \oplus \mathcal{A}'_{(j)},$$

with $\mathcal{A}'_{(j)} \simeq \mathcal{M}_{p_j} \otimes \mathbb{1}_{q_j}$ for $j \in \mathcal{J}_n$, as claimed in the statement of Theorem 4.1. Observe that we also have the minimal reducing projections for \mathcal{A}'_n which are supported on the minimal central projections P_j . For each $j \in \mathcal{J}_n$ they are the projections of rank p_j onto $\text{span}\{|j, m, \mu\rangle\}_\mu$. Thus, the explicit spatial decomposition of $\text{Fix}(\mathcal{E}_n) = \mathcal{A}'_n$ is now evident. \square

The following is a consequence of the work in this section.

Corollary 4.10. *The set of spectral projections for J^2 coincides with the set of minimal central projections for \mathcal{A}'_n and \mathcal{A}_n .*

5. Generalized Collective Rotation Channels

In this section we consider natural generalizations of collective rotation channels to higher dimensional representations of $su(2)$ (see Note 3.2). The commutation relations satisfied by the Pauli matrices are the defining properties of the Lie algebra $su(2)$. So far, we have restricted our attention to the special case where this algebra is represented by 2×2 complex matrices; specifically the Pauli matrices. Nevertheless, the algebra $su(2)$ has an irreducible representation for every integer dimension; i.e., given $d \geq 1$ it is possible to find three matrices $\Sigma_{x,d}, \Sigma_{y,d}, \Sigma_{z,d}$ of dimension d satisfying the Pauli commutation relations. Hence, the rotation group $SU(2)$ also has a representation in every integer dimension.

Note that the operators J_x, J_y, J_z , which act on 2^n -dimensional space, form a representation of the Lie group $su(2)$. But it is not an irreducible representation as the theorem in the last section shows; thus the existence of noiseless subsystems. The irreducible representations of $su(2)$ are determined by restricting these operators to a minimal reducing subspace $\mathcal{H}(j, \mu)$. Indeed, it is easily seen that these are q_j -dimensional irreducible representations of the Lie algebra $su(2)$. (The restrictions of J_x, J_y, J_z to each of these irreducible subspaces satisfies the Pauli commutation relations.)

Physicists call a d -dimensional representation of $su(2)$ a ‘spin- s ’ representation, where $d = 2s + 1$. Hence, the spin $s = \frac{d-1}{2}$ can take integer and half integer values. From this more general perspective, we see that in the previous section we considered the spin- $\frac{1}{2}$ ($d = 2$) representation of the rotation group acting on the 2-dimensional Hilbert space of a qubit. Consideration of the proof in the previous section shows that it primarily depends on the commutation relations satisfied by the generators of $su(2)$, not the particular representations of eigenvectors used

in the proof. This ‘coordinate-free’ approach allows us to readily generalize our results to collective rotation channels of arbitrary integer dimension. Most of the results from the previous section follow with small modifications, thus we shall only outline the approach.

First let us establish some notation. Let $\Sigma_{k,d}$, $k = x, y, z$, be $(2s + 1) \times (2s + 1)$ complex matrices forming an irreducible representation of $su(2)$. These matrices act on the d -dimensional Hilbert space \mathcal{H}_d of a ‘qudit’, where $d = 2s + 1$. Consider a collection of n qudits, and their associated collective rotation generators $J_{x,d}, J_{y,d}, J_{z,d}$ on $\mathcal{H}_d^{\otimes n}$, where, for instance $J_{x,d} = \sum_{k=1}^n J_{x,d}^{(k)}$ and $J_{x,d}^{(k)} = \dots \otimes \mathbb{1}_d \otimes \Sigma_{x,d} \otimes \mathbb{1}_d \dots$ with $\Sigma_{x,d}$ in the k th tensor slot. As before we may define a unital channel

$$\mathcal{E}_{n,d}(T) = E_{x,d}TE_{x,d}^\dagger + E_{y,d}TE_{y,d}^\dagger + E_{z,d}TE_{z,d}^\dagger$$

where $E_{x,d} = \exp(i\theta_x J_{x,d})$, etc. Let

$$\mathcal{A}_{n,d} = \text{Alg}\{E_{x,d}, E_{y,d}, E_{z,d}\} = \text{Alg}\{J_{x,d}, J_{y,d}, J_{z,d}\},$$

the interaction algebra for the channel. Thus the noise commutant and fixed point set coincide; $\text{Fix}(\mathcal{E}_{n,d}) = \mathcal{A}'_{n,d}$.

Proposition 5.1. *The eigenvalues of $\Sigma_{z,d}$ are $-s, -s + 1, \dots, s$, where $s = \frac{d-1}{2}$.*

Proof. This follows from the definition of $\Sigma_{z,d}$ as the restriction of the operator J_z on $\mathcal{H}(s, \mu)$. □

As in the qubit case ($s = \frac{1}{2}, d = 2$), we can thus represent a vector in $\mathcal{H}_d^{\otimes n}$ by $|\vec{i}\rangle = |i_1 i_2, \dots, i_n\rangle$ where $i_k \in \{-s, -s + 1, \dots, s\}$ denotes the eigenvalue of $\Sigma_{z,d}$ on the k th qudit. With this notation, we can restate Lemma 4.3 for arbitrary finite dimension d .

Lemma 5.2. *For $m = -sn, sn + 1, \dots, sn$ consider the subspaces of \mathcal{H}_{d^n} given by*

$$\mathcal{V}_m = \text{span}\{|\vec{i}\rangle : |\vec{i}| = m\},$$

where $|\vec{i}| = \sum_{j=1}^n i_j$. Then $\mathcal{H}_{d^n} = \sum_{-sn \leq m \leq sn} \oplus \mathcal{V}_m$ and

$$\dim \mathcal{V}_m = \sum_{k_1 + \dots + k_n = m + ns} \binom{n}{k_1 \dots k_n},$$

where $k_i \in \{0, \dots, d - 1\}$ and no repeats are allowed, even reordering, amongst the n -tuples (k_1, \dots, k_n) . Further, \mathcal{V}_m is an eigenspace for J_z corresponding to the eigenvalue m .

The proof of this Lemma follows exactly the same lines as Lemma 4.3. The analogues of Lemmas 4.4 and 4.5 also follow in a straightforward manner; they only involve the commutation relations which are independent of the representation of the algebra.

We can thus construct a basis for \mathcal{H}_{d^n} by generalizing the previous construction. The basis states are $|j, m, \mu\rangle$. The label j is for the eigenspaces of the operator

$(J^{(d)})^2 \equiv J_{x,d}^2 + J_{y,d}^2 + J_{z,d}^2$ which has eigenvalues given by $j^2 + j$ with $j \in \mathcal{J}_{n,d}$ where

$$\mathcal{J}_{n,d} = \begin{cases} \{0, 1, \dots, ns\} & \text{if } ns \text{ is an integer} \\ \{\frac{1}{2}, \frac{3}{2}, \dots, ns\} & \text{if } ns \text{ is a half integer} \end{cases}$$

The eigenspaces of $J_{z,d}$ are labelled by m , where $m = -j, -j+1, \dots, j$ (Recall that $(J^{(d)})^2$ and $J_{z,d}$ commute, so they can be simultaneously diagonalized.) Finally, μ is the extra index required to construct a basis in the common eigenspace of $(J^{(d)})^2$ and $J_{z,d}$ determined by a given pair j, m .

Let us construct these states as we did in the previous section. We start with the state $|ns, -ns, 1\rangle$ which is the unique eigenvector of $J_{z,d}$ with eigenvalue $-ns$. It is thus an eigenvector of $(J^{(d)})^2$. Then, $J_{+,d}|ns, -ns, 1\rangle$ is an eigenstate of $J_{z,d}$ with eigenvalue $-ns + 1$. Furthermore, since $J_{+,d}$ commutes with $(J^{(d)})^2$, the vectors $J_{+,d}|ns, -ns, 1\rangle$ and $|ns, -ns, 1\rangle$ are in the same eigenspace of $(J^{(d)})^2$, hence after normalizing we can label $J_{+,d}|ns, -ns, 1\rangle$ by $|ns, -ns + 1, 1\rangle$. By repeating this procedure, we find an orthonormal basis for the space

$$\begin{aligned} \mathcal{H}(ns, 1) &= \text{span}\{(J_{+,d})^p|ns, -ns, 1\rangle : 0 \leq p \leq 2ns\} \\ &= \text{span}\{|ns, m, 1\rangle : m = -ns, -ns + 1, \dots, ns\}. \end{aligned}$$

By construction, $\mathcal{H}(ns, 1)$ is a minimal reducing subspace for $\mathcal{A}_{n,d}$. Furthermore, since the spectral projections of $(J^{(d)})^2$ are the minimal central projectors of $\mathcal{A}_{n,d}$, the subspace $\mathcal{H}(ns, 1)$ is an eigenspace of $(J^{(d)})^2$.

We then consider the subspace $\mathcal{V}_{-ns+1} \ominus \text{span}\{|ns, -ns + 1, 1\rangle\}$. This is the eigenspace of $J_{z,d}$ with eigenvalue $m = -ns + 1$ which is perpendicular to the eigenspace of $(J^{(d)})^2$ labelled by ns . Hence, these vectors require a different j label, say $j = ns - 1$. We can now choose a basis for $\mathcal{V}_{-ns+1} \ominus \text{span}\{|ns, -ns + 1, 1\rangle\}$, which is labeled $|ns - 1, -ns + 1, \mu\rangle$ where the first two terms just label the subspace $\mathcal{V}_{-ns+1} \ominus \text{span}\{|ns, -ns + 1, 1\rangle\}$ and μ is an extra label to form a basis within this subspace. Thus, as we did in the previous section, we construct subspaces by applying the shift operator

$$\begin{aligned} \mathcal{H}(ns - 1, \mu) &= \text{span}\{J_{+,d}^p|ns - 1, -ns + 1, \mu\rangle : 0 \leq p \leq 2(ns - 1)\} \\ &= \text{span}\{|ns - 1, m, \mu\rangle : -ns + 1 \leq m \leq ns - 1\}. \end{aligned}$$

This procedure can be repeated with the subspaces

$$\mathcal{V}_{-m} \ominus \text{span}\{|j, -m, \mu\rangle : j = m + 1, \dots, ns, \mu = 1, \dots, q_j\}$$

to form the subspaces

$$\begin{aligned} \mathcal{H}(j, \mu) &= \text{span}\{J_{+,d}^p|j, -j, \mu\rangle : 0 \leq p \leq 2j\} \\ &= \text{span}\{|j, m, \mu\rangle : -j \leq m \leq j\}. \end{aligned}$$

The subspaces $\mathcal{H}(j, \mu)$ are minimal $\mathcal{A}_{n,d}$ -reducing and for fixed j , the subspaces $\{\mathcal{H}(j, \mu)\}_\mu$ are linked inside $\mathcal{A}_{n,d}$. Thus with this analysis in hand, we may state the following generalization of Theorem 4.1.

Theorem 5.3. *Let $\mathcal{E}_{n,d}$ be the collective rotation channel for fixed positive integers $n \geq 1$ and $d \geq 2$. Then*

$$\text{Fix}(\mathcal{E}_{n,d}) = \mathcal{A}'_{n,d} = \sum_{j \in \mathcal{J}_{n,d}} \oplus \mathcal{A}'_{(j)}, \tag{12}$$

where $\mathcal{A}'_{(j)}$ is a C^* -subalgebra of $\mathcal{A}'_{n,d}$ given, up to unitary equivalence, by

$$\mathcal{A}'_{(j)} \simeq \mathcal{M}_{p_j} \otimes \mathbb{1}_{q_j} \quad \text{for } j \in \mathcal{J}_{n,d},$$

with $p_{ns} = 1$ where and for $j \in \mathcal{J}_{n,d}$, $j < ns$,

$$p_j = \dim \mathcal{V}_j - \dim \mathcal{V}_{j+1}$$

where

$$q_j = 2j + 1 \quad \text{for } j \in \mathcal{J}_{n,d}.$$

Remark 5.4. In light of this analysis, we can extend the result to more general Lie groups. Let \mathcal{G} be a compact connected semisimple Lie group and $\mathcal{G}^{\otimes n}$ denote its n -fold tensor product. Further, let Σ_k be the set of generators of the associated Lie algebra. This algebra is entirely specified by its *structure constants* C_{kmn} defined by

$$[\Sigma_m, \Sigma_n] = i \sum_k C_{kmn} \Sigma_k. \tag{13}$$

The operators

$$J_k = (\Sigma_k \otimes \mathbb{1} \otimes \mathbb{1} \otimes \dots) + (\mathbb{1} \otimes \Sigma_k \otimes \mathbb{1} \otimes \dots) + \dots \tag{14}$$

are generators of the generalized ‘collective rotation’ which is a subgroup of $\mathcal{G}^{\otimes n}$. Clearly, they have the same structure constants as the Σ_k ; they represent the same algebra. Nevertheless, the J_k do not form an irreducible representation of the algebra. Hence, it is possible to write them as a direct sum of irreducible representations. A special property of these representations is that all the projections onto the irreducible subspaces of the same dimension are in fact ‘linked’ inside the algebra. Thus, it follows that there is an abundance of noiseless subsystems which can be explicitly identified for the corresponding quantum channels. An expansion of this analysis is contained in [19].

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