Unified and Generalized Approach to Quantum Error Correction

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We present a unified approach to quantum error correction, called *operator quantum error correction.* Our scheme relies on a generalized notion of a noiseless subsystem that is investigated here. By combining the active error correction with this generalized noiseless subsystems method, we arrive at a unified approach which incorporates the known techniques—i.e., the standard error correction model, the method of decoherence-free subspaces, and the noiseless subsystem method—as special cases. Moreover, we demonstrate that the quantum error correction condition from the standard model is a necessary condition for all known methods of quantum error correction.

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The possibility of protecting quantum information against undesirable noise has been a major breakthrough for the field of quantum computing, opening the path to potential practical applications. In this Letter, we show that the various techniques used to protect quantum information all fall under the same unified umbrella. First, we review the standard model for quantum error correction [1,2] and the passive error prevention methods of "decoherence-free subspaces" [3–5] and "noiseless subsystems" [6–8]. We then demonstrate how the latter scheme admits a natural generalization and study the necessary and sufficient conditions leading to such *generalized noiseless subsystems.* This generalized method in turn motivates a unified approach—called *operator quantum error correction*—that incorporates all aforementioned techniques as special cases. We describe this approach and discuss testable conditions that characterize when error correction is possible given a noise model. Moreover, we show that the standard error correction condition is a prerequisite for any of the known forms of error correction or prevention to be feasible.

The standard model.—What could be called the "standard model" for quantum error correction [1,2] consists of a triple $(\mathcal{R}, \mathcal{E}, C)$ where C is a subspace, a *quantum code,* of a Hilbert space $\mathcal{H}$ associated with a given quantum system. The error $\mathcal{E}$ and recovery $\mathcal{R}$ are quantum operations on $\mathcal{B}(\mathcal{H})$, the set of operators on $\mathcal{H}$, such that $\mathcal{R}$ undoes the effects of $\mathcal{E}$ on C in the following sense:

$$(\mathcal{R} \circ \mathcal{E})(\rho) = \rho \quad \text{for all } \rho = P_C \rho P_C, \tag{1}$$

where P_C is the projector of $\mathcal{H}$ onto C. As a prelude to what follows below, let us note that instead of focusing on the subspace C, we could just as easily work with the set of operators $\mathcal{B}(C)$ which act on C.

When there exists such an $\mathcal{R}$ for a given pair $\mathcal{E}, C$, the subspace C is said to be *correctable for* $\mathcal{E}$. The action of the noise operation $\mathcal{E}$ can be described in an operator-sum representation as $\mathcal{E}(\rho) = \sum_a E_a \rho E_a^\dagger$. While this decomposition is not unique, all decompositions of a given map $\mathcal{E}$ are linearly related: if $\mathcal{E}(\rho) = \sum_b F_b \rho F_b^\dagger$, then there exists scalars u_{ba} such that $F_b = \sum_a u_{ba} E_a$. We shall identify the map $\mathcal{E}$ with any of its error operators $\mathcal{E} = \{E_a\}$. The existence of a recovery operation $\mathcal{R}$ of $\mathcal{E}$ on C may be cleanly phrased in terms of the $\{E_a\}$ as follows [1,2]:

$$P_C E_a^\dagger E_b P_C = \lambda_{ab} P_C \quad \text{for all } a, b \tag{2}$$

for some scalars λ_{ab}. Clearly, this condition is independent of the operator-sum representation of $\mathcal{E}$. We note that Eq. (2) is captured as the special case of our Theorem 2 with $m = 1 = \dim \mathcal{H}^A$.

Noiseless subsystems and decoherence-free subspaces.—Let $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a quantum operation with errors $\{E_a\}$. The algebra $\mathcal{A}$ generated by the set $\{E_a, E_a^\dagger\}$ is a † algebra [9], called the *interaction algebra,* and as such it is unitarily equivalent to a direct sum of (possibly "ampliated" [9]) full matrix algebras: $\mathcal{A} \cong \bigoplus_J \mathcal{M}_{m_J} \otimes \mathbb{1}_{n_J}$. This structure induces a natural decomposition of the Hilbert space

$$\mathcal{H} = \bigoplus_J \mathcal{H}_J^A \otimes \mathcal{H}_J^B,$$

where the "noisy subsystems" $\mathcal{H}_J^A$ have dimension m_J and the "noiseless subsystems" $\mathcal{H}_J^B$ have dimension n_J.

If $\mathcal{E}$ is a *unital* quantum operation, by which we mean that the maximally mixed state $\mathbb{1}$ remains unaffected by $\mathcal{E}$ [i.e., $\mathcal{E}(\mathbb{1}) = \mathbb{1}$], then the fundamental *noiseless subsystem* (NS) method of quantum error correction [6–8] may be applied. This method makes use of the operator algebra structure of the "noise commutant,"

$$\mathcal{A}' = \{\rho \in \mathcal{B}(\mathcal{H}) : E\rho = \rho E \quad \forall\ E \in \{E_a, E_a^\dagger\}\},$$

to encode states that are immune to the errors of $\mathcal{E}$. As such, it is in effect a method of error prevention. Notice that with the structure of $\mathcal{A}$ given above, the noise commutant is unitarily equivalent to $\mathcal{A}' \cong \oplus_J \mathbb{1}_{m_J} \otimes \mathcal{M}_{n_J}$.

In [10,11] it was proved that for unital $\mathcal{E}$, the noise commutant coincides with the fixed point set for $\mathcal{E}$; i.e.,

$$\mathcal{A}' = \mathrm{Fix}(\mathcal{E}) = \{\rho \in \mathcal{B}(\mathcal{H}) : \mathcal{E}(\rho) = \rho\}.$$

This is precisely the reason that $\mathcal{A}'$ may be used to produce NS for unital $\mathcal{E}$. We note that while many of the physical noise models satisfy the unital constraint, there are important nonunital models as well. Below we show how shifting the focus from $\mathcal{A}'$ to $\mathrm{Fix}(\mathcal{E})$ (and related sets) quite naturally leads to a generalized notion of NS that applies to nonunital quantum operations as well.

For brevity, we focus on the case where information is encoded in a single noiseless sector of $\mathcal{B}(\mathcal{H})$, so

$$\mathcal{H} = (\mathcal{H}^A \otimes H^B) \oplus \mathcal{K}$$

with $\dim(\mathcal{H}^A) = m$, $\dim(\mathcal{H}^B) = n$, and $\dim\mathcal{K} = \dim\mathcal{H} - mn$. The generalization to multiple J's is straightforward. We shall write ρ^A for operators in $\mathcal{B}(\mathcal{H}^A)$ and ρ^B for operators in $\mathcal{B}(\mathcal{H}^B)$. Thus the restriction of the noise commutant $\mathcal{A}'$ to $\mathcal{H}^A \otimes \mathcal{H}^B$ consists of the operators of the form $\rho^{AB} = \mathbb{1}^A \otimes \rho^B$ where $\mathbb{1}^A$ is the identity element of $\mathcal{B}(\mathcal{H}^A)$. It is easy to see that such states are immune to noise in the unital case.

For notational purposes, assume that ordered orthonormal bases have been chosen for $\mathcal{H}^A = \mathrm{span}\{|\alpha_i\rangle\}_{i=1}^m$ and $\mathcal{H}^B = \mathrm{span}\{|\beta_k\rangle\}_{k=1}^n$ that yield the matrix representation of the corresponding subalgebra of $\mathcal{A}'$ as $\mathbb{1}_n \otimes \mathcal{M}_n$. We let $\{P_{kl} = |\alpha_k\rangle\langle\alpha_l| \otimes \mathbb{1}_n : 1 \le k, l \le m\}$ denote the corresponding family of "matrix units" associated with this decomposition. In terms of these matrix units, the *minimal reducing projectors* for $\mathcal{A}'$ are given by $P_k = |\alpha_k\rangle\langle\alpha_k| \otimes \mathbb{1}_n = P_{kk} \in \mathcal{A}$. The following equalities are readily verified and, in fact, are the defining properties for a family of matrix units.

$$P_{kl} = P_k P_{kl} P_l \quad \forall\ 1 \le k, l \le m,$$

$$P_{kl}^\dagger = P_{lk} \quad \forall\ 1 \le k, l \le m,$$

$$P_{kl}P_{l'k'} = \begin{cases} P_{kk'} & \text{if } l = l' \\ 0 & \text{if } l \neq l' \end{cases}.$$

With these properties in hand, the following useful result may be easily proved.

Lemma 1 *The map $\Gamma = \{P_{kl}\}$ from $\mathcal{B}(\mathcal{H})$ to itself satisfies the following two properties:*

$$\Gamma(\rho) = \sum_{k,l} P_{kl}\rho P_{kl}^\dagger \in \mathcal{A}', \qquad \Gamma(\rho^A \otimes \rho^B) \propto \mathbb{1}^A \otimes \rho^B,$$

for all operators ρ^A, ρ^B, and $\rho \in \mathcal{B}(\mathcal{H})$.

We note that the NS method contains the method of *decoherence-free subspaces* (DFS) [3–5] as a special case. Specifically, if we are given an error operation $\mathcal{E}$, then the DFS method encodes information in a subspace of the system's Hilbert space that is immune to the evolution. However, instead of working at the level of vectors, we could work at the level of operators. In particular, as in the standard model, we may identify a given Hilbert space $\mathcal{H}$ with the full algebra $\mathcal{B}(\mathcal{H})$ of operators acting on $\mathcal{H}$. In doing so, the DFS method may be regarded as a special case of the NS method in the sense that the DFS method in effect makes use of the "unampliated" summands, $\mathbb{1}_{m_J} \otimes \mathcal{M}_{n_J}$ where $m_J = 1$, inside the noise commutant $\mathcal{A}'$ for encoding information.

Generalized noiseless subsystems.—We now describe a generalized notion of noiseless subsystems that serves as a building block for the unified approach to error correction discussed below and applies equally well to *nonunital* maps. In the standard NS method, the quantum information is encoded in ρ^B, i.e., the state of the noiseless subsystem. Hence, it is not necessary for the noisy subsystem to remain in the maximally mixed state $\mathbb{1}^A$ under $\mathcal{E}$; it could in principle get mapped to any other state.

In order to formalize this idea, define for a fixed decomposition $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$ the set of operators

$$\mathfrak{A} = \{\rho \in \mathcal{B}(\mathcal{H}) : \rho = \rho^A \otimes \rho^B, \quad \text{for some } \rho^A \text{ and } \rho^B\}.$$

Notice that this set has the structure of a semigroup and includes operator algebras such as $\mathbb{1}^A \otimes \mathcal{B}(\mathcal{H}^B)$. For notational purposes, we assume that bases have been chosen and define the matrix units P_{kl} as above, so that $P_k = P_{kk}$, $P_{\mathfrak{A}} = P_1 + \cdots + P_m$, $P_{\mathfrak{A}}\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$, $P_{\mathfrak{A}}^\perp = \mathbb{1} - P_{\mathfrak{A}}$, and $P_{\mathfrak{A}}^\perp\mathcal{H} = \mathcal{K}$. We also define a map $\mathcal{P}_{\mathfrak{A}}$ by the action $\mathcal{P}_{\mathfrak{A}}(\cdot) = P_{\mathfrak{A}}(\cdot)P_{\mathfrak{A}}$. The following result leads to our generalized definition of NS.

Lemma 2 *Given a fixed decomposition $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B \oplus \mathcal{K}$ and a map $\mathcal{E}$, the following three conditions are equivalent, and are the defining properties of the generalized noiseless subsystem $\mathcal{H}^B$.*

(i) $\forall\ \rho^A, \rho^B\ \exists\ \sigma^A : \mathcal{E}(\rho^A \otimes \rho^B) = \sigma^A \otimes \rho^B$.

(ii) $\forall\ \rho^B\ \exists\ \sigma^A : \mathcal{E}(\mathbb{1}^A \otimes \rho^B) = \sigma^A \otimes \rho^B$.

(iii) $\forall\ \rho \in \mathfrak{A} : (\mathrm{Tr}_A \circ \mathcal{P}_{\mathfrak{A}} \circ \mathcal{E})(\rho) = \mathrm{Tr}_A(\rho)$.

Proof. The implications (i) ⇒ (ii) and (i) ⇒ (iii) are trivial. To prove (ii) ⇒ (i), observe that $\sum_{k=1}^m |\alpha_k\rangle\langle\alpha_k| = \mathbb{1}^A$, so condition (ii) implies that for any $|\psi\rangle \in \mathcal{H}^B$,

$$\sum_{k=1}^m \mathcal{E}(|\alpha_k\rangle\langle\alpha_k| \otimes |\psi\rangle\langle\psi|) = \sigma^A \otimes |\psi\rangle\langle\psi| \tag{3}$$

for some $\sigma^A \in \mathcal{B}(\mathcal{H}^A)$. Since $\mathcal{E}$ is a quantum operation, $\rho_{\psi,k} = \mathcal{E}(|\alpha_k\rangle\langle\alpha_k| \otimes |\psi\rangle\langle\psi|)$ are positive for $k = 1, \ldots, m$. Equation (3) implies that $\sigma^A \otimes |\psi\rangle\langle\psi|$ is a convex combination of the operators $\rho_{\psi,k}$, which is possible only if $\rho_{\psi,k} = \rho^A_{\psi,k} \otimes |\psi\rangle\langle\psi|$ for some positive $\rho^A_{\psi,k}$. Through an application of the Stinespring dilation theorem [12] and a linearity argument, it follows that $\rho^A_{\psi,k}$ does not depend on ψ. Since the basis $\{|\alpha_k\rangle\}$ and the state $|\psi\rangle$ were chosen arbitrarily, the result now follows from the linearity of $\mathcal{E}$.

To prove (iii) ⇒ (ii), note that since $\mathcal{E}$ and Tr_B are trace preserving, (iii) implies that $(\mathcal{P}_{\mathfrak{A}} \circ \mathcal{E})(\rho) = \mathcal{E}(\rho)$ for all $\rho \in \mathfrak{A}$. By setting $\rho = \mathbb{1}^A \otimes |\psi\rangle\langle\psi|$ as above, we conclude from (iii) that $\mathcal{E}(\rho) = \sigma^A \otimes |\psi\rangle\langle\psi|$ for some σ^A. The rest follows from linearity. □

The subsystem $\mathcal{H}^B$ is said to be *noiseless* when it satisfies one—and hence all—of the conditions in Lemma 2. It is clear from the third condition that the fate of the noisy subsystem $\mathcal{H}^A$ has no importance: only the information stored in the noiseless subsystem $\mathcal{H}^B$ must be preserved by $\mathcal{E}$. Note that the generalized definition of NS coincides with the standard definition when $\dim(\mathcal{H}^A) = 1$. Hence, the notion of DFS is not altered by this generalization.

Given this new notion of a NS, the crucial question is to determine what are the necessary and sufficient conditions for a map $\mathcal{E} = \{E_a\}$ to admit a NS described by a semigroup $\mathfrak{A}$. Recall that the condition expressed by Eq. (2) gives an answer for standard error correction. The following theorem provides an answer to this question in the general noiseless subsystem setting.

Theorem 1 *Let $\mathcal{E} = \{E_a\}$ be a quantum operation on $\mathcal{B}(\mathcal{H})$ and let $\mathfrak{A}$ be a semigroup in $\mathcal{B}(\mathcal{H})$ as above. Then $\mathfrak{A}$ encodes a noiseless subsystem (decoherence-free subspace in the case $m = 1$)—as defined by any of the three conditions of Lemma 2—if and only if the following two conditions hold:*

$$P_k E_a P_l = \lambda_{akl} P_{kl} \quad \text{for all } a, k, l \tag{4}$$

for some set of scalars $\{\lambda_{akl}\}$ and

$$P_{\mathfrak{A}}^{\perp} E_a P_{\mathfrak{A}} = 0 \quad \text{for all } a. \tag{5}$$

Proof. To prove the necessity of Eqs. (4) and (5), note that Lemma 1 and Lemma 2 imply

$$(\Gamma \circ \mathcal{E} \circ \Gamma)(\rho) \propto \Gamma(\rho) \quad \text{for all } \rho \in \mathcal{B}(\mathcal{H}). \tag{6}$$

By linearity, the proportionality factor cannot depend on ρ, so the sets of operators $\{P_{ki} E_a P_{jl}\}$ and $\{\lambda P_{k'l'}\}$ define the same map for some scalar λ. We may thus find a set of scalars $\mu_{kiajl,k'l'}$ such that

$$P_{ki} E_a P_{jl} = \sum_{k'l'} \mu_{kiajl,k'l'} P_{k'l'}. \tag{7}$$

Multiplying both sides of this equality on the right by P_l and on the left by P_k, we see that $\mu_{ijakl,i'l'} = 0$ when $k \neq k'$ or $l \neq l'$. This implies Eq. (4) with $\lambda_{akl} = \mu_{kkall,kl}$.

For the second condition, note that by definition $P_{\mathfrak{A}}^{\perp} \rho P_{\mathfrak{A}}^{\perp} = 0$ for all $\rho \in \mathfrak{A}$. Together with Lemma 1 and Lemma 2, this implies $P_{\mathfrak{A}}^{\perp} \mathcal{E}(\Gamma(\rho)) P_{\mathfrak{A}}^{\perp} = 0$ for all $\rho \in \mathcal{B}(\mathcal{H})$. Equation (5) follows from this observation via a consideration of the operator-sum representation for $\mathcal{E}$.

To prove sufficiency, we use the definitions $\mathbb{1} = P_{\mathfrak{A}} + P_{\mathfrak{A}}^{\perp}$ and $P_{\mathfrak{A}} = \sum_{k=1}^{m} P_k$ to establish for all $\rho \in \mathfrak{A}$

$$\begin{aligned}\mathcal{E}(\rho) &= (P_{\mathfrak{A}} + P_{\mathfrak{A}}^{\perp}) \sum_a E_a \rho E_a^{\dagger} (P_{\mathfrak{A}} + P_{\mathfrak{A}}^{\perp}) \\ &= \sum_a P_{\mathfrak{A}} E_a \rho E_a^{\dagger} P_{\mathfrak{A}} = \sum_{a,k,k'} P_k E_a \rho E_a^{\dagger} P_{k'}.\end{aligned}$$

Given $\rho = \rho^A \otimes \rho^B \in \mathfrak{A}$, we have

$$\rho^A \otimes \rho^B = P_{\mathfrak{A}} (\rho^A \otimes \rho^B) P_{\mathfrak{A}} = \sum_{l,l'} P_l (\rho^A \otimes \rho^B) P_{l'}.$$

Combining this with the above identity implies

$$\begin{aligned}\mathcal{E}(\rho^A \otimes \rho^B) &= \sum_{a,k,k',l,l'} P_k E_a P_l (\rho^A \otimes \rho^B) P_{l'} E_a^{\dagger} P_{k'} \\ &= \sum_{a,k,k',l,l'} \lambda_{akl} \overline{\lambda}_{ak'l'} P_{kl} (\rho^A \otimes \rho^B) P_{l'k'}.\end{aligned}$$

The proof now follows from the fact that the matrix units P_{kl} act trivially on the $\mathcal{B}(\mathcal{H}^B)$ sector. □

Conditions Eqs. (4) and (5) do not necessarily imply that the noiseless operators are in the commutant of the interaction algebra $\mathcal{A} = \{E_a\}$ since $P_{\mathfrak{A}} E_a P_{\mathfrak{A}}^{\perp}$ is not necessarily equal to zero. Hence, this generalization does, indeed, admit new possibilities.

The unified approach.—The unified scheme for quantum error correction consists of a triple $(\mathcal{R}, \mathcal{E}, \mathfrak{A})$ where again $\mathcal{R}$ and $\mathcal{E}$ are quantum operations on some $\mathcal{B}(\mathcal{H})$, but now $\mathfrak{A}$ is a semigroup in $\mathcal{B}(\mathcal{H})$ defined as above with respect to a fixed decomposition $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$. Given such a triple $(\mathcal{R}, \mathcal{E}, \mathfrak{A})$ we say that $\mathfrak{A}$ is *correctable for $\mathcal{E}$* if

$$(\mathrm{Tr}_A \circ \mathcal{P}_{\mathfrak{A}} \circ \mathcal{R} \circ \mathcal{E})(\rho) = \mathrm{Tr}_A(\rho) \quad \text{for all } \rho \in \mathfrak{A}. \tag{8}$$

In other words, $(\mathcal{R}, \mathcal{E}, \mathfrak{A})$ is a correctable triple if the $\mathcal{H}^B$ sector of the semigroup $\mathfrak{A}$ encodes a noiseless subsystem of the error map $\mathcal{R} \circ \mathcal{E}$. Thus, substituting $\mathcal{E}$ by $\mathcal{R} \circ \mathcal{E}$ in Lemma 2 offers alternative equivalent definitions of a correctable triple. Observe that the standard model for error correction is given by the particular case in this model that occurs when $m = 1$. Lemma 2 shows that the generalized (and standard) NS and DFS methods are captured in this model when $\mathcal{R} = \mathrm{id}$ is the identity channel and, respectively, $m \geq 1$ and $m = 1$.

We next present a mathematical condition that characterizes correctable codes for a given channel $\mathcal{E}$ in terms of its error operators and generalizes Eq. (2) for the standard model. Again, we assume that matrix units P_{kl} associated with the noise commutant have been defined as above.

Theorem 2 *Let $\mathcal{E} = \{E_a\}$ be a quantum operation on $\mathcal{B}(\mathcal{H})$ and let $\mathfrak{A}$ be a semigroup in $\mathcal{B}(\mathcal{H})$ as above. If $\mathfrak{A}$ is* correctable for *$\mathcal{E}$ [see Eq. (8)], then there are scalars $\Lambda = \{\lambda_{abkl}\}$ such that*

$$P_k E_a^{\dagger} E_b P_l = \lambda_{abkl} P_{kl} \quad \text{for all } a, b, k, l. \tag{9}$$

Proof. As noted above $(\mathcal{R}, \mathcal{E}, \mathfrak{A})$ being a correctable triple implies that $\mathfrak{A}$ encodes a generalized noiseless subsystem of the map $\mathcal{R} \circ \mathcal{E}$. Applying Theorem 1, and, in particular, condition Eq. (4), to the map $\mathcal{R} \circ \mathcal{E}$ implies the existence of a set of scalars μ_{cakl} for which $P_k R_c E_a P_l = \mu_{cakl} P_{kl}$. It now follows from Eq. (5) applied to the map $\mathcal{R} \circ \mathcal{E}$ and $P_{\mathfrak{A}} = \sum_j P_j$ that

$$\begin{aligned} P_k E_a^\dagger E_b P_l &= \sum_c P_k^\dagger E_a^\dagger R_c^\dagger R_c E_b P_l \\ &= \sum_{c,j} P_k^\dagger E_a^\dagger R_c^\dagger P_j^\dagger P_j R_c E_b P_l \\ &= \sum_{c,j} \overline{\mu}_{cajk} \mu_{cbjl} P_{jk}^\dagger P_{jl} = \Big(\sum_{c,j} \overline{\mu}_{cajk} \mu_{cbjl} \Big) P_{kl}, \end{aligned}$$

and this completes the proof of the Theorem. □

Remark 1 *The condition Eq.* (9) *is independent of the choice of basis* $\{|\alpha_i\rangle\}$ *that defines the family* P_{kl} *and of the operator-sum representation of* $\mathcal{E}$. *In particular, under the changes* $|\alpha_k'\rangle = \sum_l u_{kl} |\alpha_l\rangle$ *and* $F_a = \sum_b w_{ab} E_b$, *the scalars* Λ *change to* $\lambda'_{abkl} = \sum_{a'b'k'l'} \overline{u}_{kk'} u_{l'l} \overline{w}_{aa'} w_{bb'} \lambda_{abkl}$.

Equation (9) generalizes the quantum error correction condition Eq. (2) to the case where information is encoded in operators, not necessarily restricted to act on a fixed code subspace C. However, observe that setting $k = l$ in Eq. (9) gives the standard error correction condition Eq. (2) with $P_C = P_k$. This leads to the following result.

Theorem 3 *If* $(\mathcal{R}, \mathcal{E}, \mathfrak{A})$ *is a correctable triple for some semigroup* $\mathfrak{A}$ *defined as above, then* $(\mathcal{P}_k \circ \mathcal{R}, \mathcal{E}, P_k \mathfrak{A}(P_k))$ *is a correctable triple according to the standard definition Eq.* (2), *where* P_k *is any minimal reducing projector of* $\mathfrak{A}$, *and the map* $\mathcal{P}_k$ *is defined by* $\mathcal{P}_k(\cdot) = \sum_l P_{kl}(\cdot) P_{kl}^\dagger$.

Proof. The error correction condition Eq. (8) and Lemma 2 imply that for all ρ^B there is a σ^A such that

$$(\mathcal{R} \circ \mathcal{E})(P_k(\mathbb{1}^A \otimes \rho^B) P_k) \propto \sigma^A \otimes \rho^B.$$

Observe that $\mathcal{P}_k(\sigma^A \otimes \rho^B) \propto |\alpha_k\rangle\langle\alpha_k| \otimes \rho^B$ for all ρ^B and σ^A. Combining these two observations, we conclude that

$$(\mathcal{P}_k \circ \mathcal{R} \circ \mathcal{E})(P_k(\mathbb{1}^A \otimes \rho^B) P_k) \propto P_k(\mathbb{1}^A \otimes \rho^B) P_k,$$

completing the proof. □

Theorem 3 has important consequences. Given a map $\mathcal{E}$, the existence of a correctable code subspace C—captured by the standard error correction condition Eq. (2)—is a prerequisite to the existence of any known type of error correction or prevention scheme (including the generalizations introduced in this Letter). Moreover, Theorem 3 shows how to transform any one of these error correction or prevention techniques into a standard error correction scheme.

Finally, note that Theorem 2 sets necessary conditions for the possibility of operator quantum error corrections, but does not address sufficiency. At the time of writing, we have not proved sufficiency in full generality. We have, however, demonstrated that these conditions are sufficient for a number of motivating special cases. This topic will be discussed in an upcoming paper.

Conclusion.—We have presented a general model for quantum error correction, called *operator quantum error correction,* that unifies the fundamental paradigms. In doing so, we have generalized the method of *active* error correction by implementing the condition at the level of operators rather than subspaces. We have also generalized the notion of noiseless subsystems by relaxing the constraints imposed on the "noisy" sector of the algebra, i.e., that it remains in the maximally mixed state. In addition, we have demonstrated that the standard error condition Eq. (2) is a necessary condition for any type of error correction—either passive or active—to be possible, and we have shown how to convert any such scheme into a standard error correction protocol.

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