Local topological order inhibits thermal stability in 2D

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We study the robustness of quantum information stored in the degenerate ground space of a local, frustration-free Hamiltonian with commuting terms on a 2D spin lattice. On one hand, a macroscopic energy barrier separating the distinct ground states under local transformations would protect the information from thermal fluctuations. On the other hand, local topological order would shield the ground space from static perturbations. Here we demonstrate that local topological order implies a constant energy barrier, thus inhibiting thermal stability.

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A self-correcting quantum memory [1] is a physical system whose quantum state can be preserved over a long period of time without the need for any external intervention. The archetypal self-correcting classical memory is the two-dimensional (2D) Ising ferromagnet. The ground state of this system is two-fold degenerate—all-spin up and all-spin down—so it can store one bit of information. If the memory is put into contact with a heat bath after being initialized in one of these ground states, thermal fluctuations will lead to the creation of small error droplets of inverted spins. The boundary of such droplets are domain walls, i.e., one-dimensional excitations whose energy is proportional to the droplet perimeter. If the temperature is below the critical Curie temperature, the Boltzman factor will prevent the creation of macroscopic error droplets. Thus, when the system is cooled down (either physically or algorithmically) after some macroscopic storage time, it will very likely return to its original ground state: the memory is thermally stable.

This behaviour contrasts with the 1D Ising ferromagnet whose domain walls are point-like excitations. Therefore, they can freely diffuse on the chain at no energy cost. As a consequence, arbitrarily large error droplets can form, so this 1D memory is thermally unstable.

While the 2D Ising ferromagnet features thermal stability, it is vulnerable to static, local perturbations. Indeed, an arbitrarily weak magnetic field breaks the ground state degeneracy and favours one ground state over the other. When this perturbed system is subject to thermal fluctuations, the bulk contribution of the magnetic field overwhelms the boundary tension of the domain wall, so once error droplets reach a critical size, they rapidly expand to corrupt the memory. This type of instability plagues any systems with a local order parameter, so they cannot be robust quantum memories. Indeed, distinct ground states give different values of this order parameter, so a local field coupling to the order parameter lifts degeneracy.

In 2D and higher, there exists quantum systems with no local order parameter and whose spectrum is stable under weak, local perturbations. These systems have a degenerate ground state separated from the other energy levels by a constant energy gap, and perturbations only alter these features by an exponentially vanishing amount as a function of the system size. Kitaev’s toric code [2] is the best known example of this type. However, excitations in Kitaev’s code are point-like objects—as for the 1D Ising model—so it does not offer a macroscopic energy barrier protection to thermal fluctuations [1,3-6].

In this Letter, we study the possibility of combining the thermal stability of the 2D Ising model with the spectral stability of Kitaev’s code to obtain a robust quantum memory in 2D. We consider $d$-level spins located at the vertices $V$ of a 2D lattice $\Lambda = (V,E)$, with Hamiltonian

$$H = -\sum_{X \subset V} P_X,$$

where $P_X$ is supported on the subset $X$ of the spins, i.e., it acts trivially on the complement $X = V - X$ of $X$. The Hamiltonian is local in the sense that $P_X = 0$ whenever $X$ has radius larger than some constant $w$. Since we are only interested in the ground state and scaling of the energy gap, we can assume without loss of generality that each $P_X$ is a projector. We also assume that $H$ is frustration-free, meaning that the ground states minimize the energy of each term of the Hamiltonian separately, i.e., $P_X|\psi_0\rangle = |\psi_0\rangle$. Then, the ground space $C$ is the image of the code projector $P = \prod_X P_X$ (henceforth, the $P_X = 0$ are not included in such products).

This family of lattice models, called local commuting projector code (LCPC) includes most models of topological order, e.g. [2,7,8]. It has been proved that LCPC have a stable spectrum [3,11] if they obey the following local topological order condition.

Definition 1 (Local topological order). For any topologically trivial region $A$, let $P_A = \prod_{X : X \cap A \neq \emptyset} P_X$ be the product of projectors that intersect region $A$. For a system with local topological order, the density matrices $\rho_A = Tr_A P$ and $\rho_{loc} = Tr_\Lambda P_A$ have the same kernel and $\rho_A$ is proportional to a projector.

Our main result is that any system with local topological order has only a constant energy barrier between
ground states, similar to Kitaev’s code. In fact, our proof is inspired by the behaviour of Kitaev’s code: we exhibit a procedure creating a point-like defect at one boundary of the lattice and dragging it across to another boundary, at a constant energy cost. The difficulty stems from the fact that, as noted by Haah and Preskill [12], it is unclear whether this process can be realized unitarily in general.

The rest of the Letter is organized as follows. We first introduce a few definitions and review known facts about LCPCs. We then present a local noise model that requires a constant amount of energy and demonstrate that (i) it can be executed in a time that scales linearly with system size for any system with local topological order, and (ii) it corrupts the information content of the memory. We end with a discussion and general conclusions.

Background— Characterizing the thermal stability of a memory requires detailed knowledge of its thermalization process. Since we seek to address a broad class of systems, our analysis cannot be model specific. We thus retain only two essential features common to all thermalization processes: (i) the bath interacts locally with the system, and (ii) high-energy states are penalized. As we now explain, we can combine these features to obtain a sufficient condition for thermal stability.

We will say that a memory with Hamiltonian Eq. (1) has an energy barrier at most \( \Delta \) if there exists a ground state \( \psi_0 \) and a sequence of \( T \in \text{poly}(N) \) CPTP maps \( \mathcal{E}_k \), each acting locally on the system and a finite-dimensional ancilla \( A \), such that (i) starting from \( \psi_0 \), the sequence returns the system to the ground space, (ii) in a state that differs from the initial state \( \psi_0 \), and (iii) the energy of any intermediate state is at most \( \Delta \) above the ground state energy \( E_0 \). More formally, these conditions are

\[
\text{Tr} [P \cdot \mathcal{E}_T \ldots \mathcal{E}_2 \mathcal{E}_1 (\psi_0 \otimes \rho_A)] \geq \frac{2}{3} \tag{2}
\]

\[
\text{Tr} [\psi_0 \cdot \mathcal{E}_T \ldots \mathcal{E}_2 \mathcal{E}_1 (\psi_0 \otimes \rho_A)] \leq \frac{1}{3} \tag{3}
\]

\[
\max_{k \in 1, 2, \ldots, T} \text{Tr} [H \cdot \mathcal{E}_k \ldots \mathcal{E}_2 \mathcal{E}_1 (\psi_0 \otimes \rho_A)] - E_0 = \Delta \tag{4}
\]

where \( P \) is the code projector, i.e., projector onto the ground space of \( H \), and the factors \( \frac{2}{3} \) and \( \frac{1}{3} \) are arbitrarily chosen constants. The additional ancillary system, initially in state \( \rho_A \), is used to model some finite non-Markovian effects of the bath, so each map \( \mathcal{E}_k \) has complete access to it. The energy barrier of a memory is taken to be the smallest value of \( \Delta \) over all such sequences of maps. If a memory has a macroscopic energy barrier \( \Delta \geq N^\alpha \) for some constant \( \alpha > 0 \), then any short sequence of local transformations that returns the system to an altered ground state must visit a high energy state, and is therefore thermally stable. Our main result is obtained by exhibiting a sequence of maps \( \mathcal{E}_k \) with an energy barrier \( \Delta \) that is a constant, independent of the system size \( N \). In addition, we show that the length \( T \) of this sequence is proportional to the linear size of the lattice when the system has local topological order.

We call logical operator an operator \( L \) that maps the ground space to itself, i.e., \( [L, P] = 0 \) or equivalently \( [L, P_X] = 0 \) for all \( X \). In particular, we are interested in logical operators that act non-trivially on the code space, i.e., \( LP \neq P \), as they can alter the encoded information. In a series of paper [12]–[15], it was shown that LCPCs always admit at least one non-trivial logical operator supported only on a 1D (constant width) strip of the lattice.

An important subclass of LCPCs are stabilizer codes [14], for which \( P_X = \frac{1}{2} (I + S_X) \) with \( S_X \) tensor-product of Pauli matrices \( \sigma_{0,1,2,3} \) (with \( \sigma_0 \) being the identity \( I \)). Because of this particular structure, the non-trivial string-like logical operator described above is also a tensor product of Pauli matrices, \( L = \otimes^k_{\ell} \sigma^k_{\ell} \) where \( k \) labels the \( \ell \) sites along the strip in some natural way, from left to right, say. Then, applying the error sequence \( \{ \sigma^k_{\ell} \} \) will build up to the operator \( L \), and will only visit intermediate states with a constant energy above the ground state. Indeed, at an intermediate state \( 0 < n < \ell \), only a segment \( \otimes^k_{\ell} \sigma^k_{\ell} \) of the logical operator \( L \) has been applied. This segment commutes with all terms \( P_X \) except the ones within distance \( O(w) \) from site \( k \), so only these terms contribute to the energy: the excitations are point-like objects located in the vicinity on the end of the string segment, so the memory is unstable [13]–[15] [17]. This simple argument fails for more general LCPCs because logical operators do not have a tensor product structure.

Noise model— In this Section we present an error sequence \( \{ \mathcal{E}_k \} \) that achieves a constant energy barrier. To motivate our construction, let us first consider a simplified version. In a first step, all particles on the strip are removed and replaced by particles in random states. Mathematically, this action can be described by the maximally depolarizing channel \( \mathcal{D}_k \) on all the sites \( k \) of the strip, which is on average equivalent to applying a Haar-random unitary operator on each site \( k \),

\[
\mathcal{D}_k[\psi] = \text{Tr}_k [\psi] \otimes I_k / D = \int U_k |\psi\rangle \langle U_k| dU_k. \tag{5}
\]

At this point, it is clear (via no-cloning) that the information content of the strip has been wiped out, and that it cannot be recovered by any subsequent operation. To return the system to its ground space, we can measure the projector \( P \), and, with lots of luck, obtain the result \(+1\). Of course this outcome is extremely unlikely, and for this reason we will refer to this noise model as the fortuitous model. Note that prior to the measurement of \( P \), we can apply any CPTP map on the strip without altering our main conclusion that the information has been erased.

Clearly, the fortuitous model is not what we are seeking. Not only does it rely on an extraordinary amount of luck, but the state after the first step has energy propor-
The sequential noise model interleaves the depolarizing and projection steps of the fortuitous model to obtain individual iterations for every site $k \in L$, which succeed with high probability. Each iteration consists of several trials. Trial $m$ of iteration $k$ corresponds to (i) applying a trial unitary $U_k^{(m)}$ on site $k$, chosen at random from the Haar measure, and (ii) measuring the local constraint $P_{k-1,k}$. Trials are repeated until a successful trial in which the +1 outcome of $P_{k-1,k}$ is obtained and the next iteration begins. Given the state on the strip, a unitary is eligible if it leads to a successful trial with non-zero probability. The initial iteration $k = 1$ differs since the constraint is not measured. Physically, the whole procedure corresponds to creating a random excitation at iteration 1, and moving it along the strip across to the opposite edge by subsequent iterations.

The sequential noise model only creates intermediate states of constant energy. The reason is the same as for stabilizer codes: the excitations are point-like objects. Indeed, during iteration $k$, the state is almost everywhere indistinguishable from a ground state because it obeys all constraints $P_{i-1,i}$, except $P_{k-1,k}$ and $P_{k,k+1}$ since only those potentially do not commute with $U_k^{(m)}$. Furthermore, a failed trial during iteration $k$ does not affect the outcome of previous iterations since local constraints commute. Thus, we only need to show that the expected number of trials at each iteration is independent of lattice size—so the total duration of the noise process grows linearly with the lattice size $\ell$—and that the average effect of a complete sequence of $\ell$ successful iterations has exactly the same effect as the fortuitous model.

**Expected number of trials**— The sequential model would run into a dead-end, an iteration requiring an infinite number of trials, if the state of the strip admits no eligible unitary at the $k^{th}$ iteration [27]. Such a dead-end occurs in the Ising-like toric code introduced in [9], where the plaquette operators $B_p$ of the toric code are replaced by Ising-like interaction $B_pB_q$ whose symmetry is broken by a single defect plaquette $B_{p'}$ to recover the toric code ground state. The sequential model could start preparing the $B_p = -1$ sector and reach a dead-end when it encounters the $B_{p'} = +1$ constraint. However, that code does not have local topological order, and its spectrum is indeed unstable ($B_{p'}$ is a local order parameter). We now show that such dead-ends do not occur with local topological order.

**Proposition 2.** Local topological order implies that, at any iteration $k$, there exists an eligible unitary $U_k$.

**Proof.** We will prove the contrapositive. Let $\psi$ be the state during the $k^{th}$ iteration for which no $U_k$ is eligible. We have

$$P_{k-1,k}U_k|\psi\rangle = 0 \quad \forall U_k,$$

(6)

which average over the Haar measure, using Eq. (5), is

$$P_{k-1,k}(\text{Tr}_k[|\psi\rangle \langle \psi|] \otimes I_k/D) = 0.$$  

(7)

Tracing out the region $R_k = \{(i,j) : i \geq k \text{ and } |j| \leq 1\} \subset L'$ of the extended strip located at the right of site $k$, c.f. Fig. 1, Eq. (7) yields

$$\text{Tr}_k[P_{k-1,k}\text{Tr}_{R_k}[|\psi\rangle]} = 0.$$  

(8)

Thus, there exists a state $|\xi\rangle$ in the support of $\text{Tr}_{R_k}[|\psi\rangle]$ which is in the image of $P_{l-1,i}$ for $i < k$ but also is in the kernel of $\text{Tr}_k[P_{k-1,k}]$. This entails violation of local topological order on site $k - 2$ since $\text{Tr}_k[|\xi\rangle] \in$ the kernel of $\rho_{k-2} = \text{Tr}_{k-2}[P_{k-3,k-2}P_{k-2,k-1}] = 0$. 

**Proposition 3.** When the system has local topological order, the expected number of trials $A_k$ at iteration $k$ is finite and independent of the system size.

**Proof.** We introduce two maps

$$P_{k-1,k}[\rho] = P_{k-1,k}\rho P_{k,k+1}$$

$$Q_{k-1,k}[\rho] = (I - P_{k-1,k})\rho(I - P_{k-1,k})$$

(9)

which represent a successful and failed measurement of the local constraint $P_{k-1,k}$. In a failed trial, the map $Q_{k-1,k}$ is always immediately preceded and followed by
a depolarization of site \( k \). This sequence can be rewritten in an equivalent form, c.f. Fig. 2

\[
\mathcal{E}_{k-1} \otimes D_k = D_k \mathcal{Q}_{k-1,k} D_k \tag{11}
\]

which defines a biasing map \( \mathcal{E}_{k-1} \). This map is not trace-preserving since the trace of its unnormalized output state is the average probability of a failed trial.

The sequence of \( m \) failed trials followed by a successful trial produces the map

\[
\mathcal{P}_{k-1,k} D_k (\mathcal{Q}_{k-1,k} D_k)^m = \mathcal{P}_{k-1,k} (\mathcal{E}_{k-1}^m \otimes D_k) \tag{12}
\]

where we have used Eq. \((11)\) and \( D_k^2 = D_k \). Thus, given a state \( \psi \), the average probability \( p_k^{(m)}(\psi) \) of a success after \( m \) failures is \( p_k^{(m)}(\psi) = \text{Tr} \left[ \mathcal{P}_{k-1,k} (\mathcal{E}_{k-1}^m \otimes D_k) [\psi] \right] \). Therefore, the expected number of trials

\[
A_k(\psi) = \sum_{m=1}^{\infty} (m+1) \text{Tr} \left[ \mathcal{P}_{k-1,k} (\mathcal{E}_{k-1}^m \otimes D_k) [\psi] \right]
\]

\[
= \text{Tr} \left[ \mathcal{P}_{k-1,k} \left( (I - \mathcal{E}_{k-1})^{-1} \otimes D_k \right) [\psi] \right]
\]

is bounded by the norm of the superoperator inside the trace and thus only depends on the microscopic details of \( H \), not on system size. Note that the geometric sum of Eq. \((14)\) converges since \( \mathcal{E}_{k-1} \) cannot have \(+1\) eigenvectors in the groundspace for a local topological ordered system, since those would contradict Proposition 2.

**Equivalence between models**—We now prove the equivalence of the sequential and fortuitous models. The effect of iteration \( k \) averages to the map

\[
K_{k-1,k} = \sum_{m=0}^{\infty} \mathcal{P}_{k-1,k} (\mathcal{E}_{k-1}^m \otimes D_k)
\]

\[
= \mathcal{P}_{k-1,k} \left( (I - \mathcal{E}_{k-1})^{-1} \otimes D_k \right),
\]

so the total action of the sequential noise model is

\[
K = \prod_{k=2}^{L} \mathcal{P}_{k-1,k} \left( (I - \mathcal{E}_{k-1})^{-1} \otimes D_k \right) \mathcal{D}_1.
\]

Terms with non-overlapping support trivially commute. We thus move all depolarizing channels to act first, which globally depolarizes the strip. To move the biasing operators past the projectors, it suffices to prove that \( C = [\mathcal{E}_k, \mathcal{P}_{k-1,k}] \mathcal{D}_{k+1} = 0 \) since \( \mathcal{Q}_{k,k+1} \) and \( \mathcal{P}_{k-1,k} \) commute. Hence, the terms of Eq. \((17)\) can be reordered into

\[
K = \prod_{k=2}^{L} \mathcal{P}_{k-1,k} \prod_{k=2}^{L+1} (I - \mathcal{E}_{k-1})^{-1} \prod_{k=1}^{L} \mathcal{D}_k \tag{18}
\]

which has the form of the fortuitous model: complete depolarization of the strip, arbitrary transformation on the strip, and projection onto the ground space. 

**Discussion**—While we have focused on systems with open boundary conditions, our results extend straightforwardly to periodic boundaries. The noise process could begin at some arbitrary location where it would create a pair of defects and, using the techniques we presented, wrap one of these defects around the system. A final measurement would then attempt to fuse both defects back into the ground space. Note that it may be necessary to wrap one defect several times around the system to obtain a successful fusion. For instance, Kitaev’s code with a twisted boundary requires two complete wraps.

The noise process we presented corrupts the memory after a time that grows proportionally to the system size, which can be interpreted as a (macroscopic) upper bound to the storage time. However, there are good reasons to believe that the actual storage time is in fact independent of the system size. At nonzero temperature we expect a finite density of defects, so the noise process we described could be happening in parallel all over the lattice. As pairs of defects meet, they can fuse to the vacuum with some probability to create longer error strings. The memory time is then related to the percolation of these error chains, which should be independent of the system size.

Our result does not completely close the door to the existence of a robust quantum memory in 2D. First, local topological order is a sufficient, but perhaps not necessary condition for spectral stability. Systems with a stable spectrum that do not have local topological order would escape our conclusions. Second, we have restricted the form of the Hamiltonian. In realistic physical systems, the terms \( P_X \) need not to commute, the ground space can be frustrated, and interaction can decay algebraically with the distance between sites. Third, a macroscopic energy barrier is one mechanism that leads to thermal stability, but they may exist other mechanism. In particular, a system in contact with a heat bath tends to minimize its free energy \( F = E - TS \). Thus, we could imagine a system with a large entropy barrier: among all possible local noise sequences, only a vanishingly small fraction can take one ground state to another, while the overwhelming majority lead to dead-ends as described above. Such topological spin-glasses \[19\] could offer an enhanced quantum memory lifetime. This proposal is distinct from existing studies showing that disorder induces an exponential localization of anyons \[20\] \[21\], as those only address zero-temperature storage.
Conclusion—Our main result hints at a general trade-off in 2D between a quantum memory’s ability to suppress thermal noise and its stability to static perturbations. Recent discoveries [24, 26] show that this tradeoff is not necessary in 3D. Our result extends prior findings [13, 15] derived for stabilizer codes to a broader, widely studied class of models that includes quantum double [2], Levin-Wen [7], and Turavn-Viro [8] models among others.

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[27] Note that while the eligibility of a unitary depends on the state in general, it cannot change if the state has only support on the kernel of $P_{k−1,k}$. 