Local Topological Order Inhibits Thermal Stability in 2D

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We study the robustness of quantum information stored in the degenerate ground space of a local, frustration-free Hamiltonian with commuting terms on a 2D spin lattice. On one hand, a macroscopic energy barrier separating the distinct ground states under local transformations would protect the information from thermal fluctuations. On the other hand, local topological order would shield the ground space from static perturbations. Here we demonstrate that local topological order implies a constant energy barrier, thus inhibiting thermal stability.

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A self-correcting quantum memory [1] is a physical system whose quantum state can be preserved over a long period of time without the need for any external intervention. The archetypical self-correcting classical memory is the 2D Ising ferromagnet. The ground state of this system is twofold degenerate, and so it can store one bit of information. If the memory is put into contact with a heat bath after being initialized in one of these ground states, thermal fluctuations will lead to the creation of small error droplets of inverted spins. The boundary of such droplets are domain walls, i.e., one-dimensional excitations whose energy is proportional to the droplet perimeter. If the temperature is below the critical Curie temperature, the Boltzmann factor will prevent the creation of macroscopic error droplets. Thus, when the system is cooled down (either physically or algorithmically) after some macroscopic storage time, it will very likely return to its original ground state: the memory is thermally stable.

This behavior contrasts with the 1D Ising ferromagnet whose domain walls are pointlike excitations. The creation of a domain wall costs some constant amount of energy (the gap), but once created the excitations can freely diffuse on the chain at no extra energy cost. As a consequence, arbitrarily large error droplets can form at a constant energy cost, so this 1D memory is thermally unstable.

While the 2D Ising ferromagnet features thermal stability, it is vulnerable to static, local perturbations. Indeed, an arbitrarily weak magnetic field breaks the ground state degeneracy and favors one ground state over the other. When this perturbed system is subject to thermal fluctuations, the bulk contribution of the magnetic field overwhelms the boundary tension of the domain wall, so that once error droplets reach a critical size they rapidly expand to corrupt the memory. This type of instability plagues any system with a local order parameter, and so they cannot be robust quantum memories. Indeed, distinct ground states give different values of this order parameter, so that a local field coupling to the order parameter lifts degeneracy.

In 2D and higher, there exist quantum systems with no local order parameter and whose spectrum is stable under weak, local perturbations. These systems have a degenerate ground state separated from the other energy levels by a constant energy gap, and perturbations only alter these features by an exponentially vanishing amount as a function of the system size. Kitaev's toric code [2], a $\mathbb{Z}_2$ spin liquid, is the best known example of this type. However, excitations in Kitaev's code are pointlike objects—as for the 1D Ising model—and so it does not offer a macroscopic energy barrier protection to thermal fluctuations [1,3–6].

In this Letter, we study the possibility of combining the thermal stability of the 2D Ising model with the spectral stability of Kitaev's code to obtain a robust quantum memory in 2D. We consider d-level spins located at the vertices V of a 2D lattice $\Lambda = (V, E)$, with Hamiltonian

$$H = -\sum_{X \subset V} P_X, \text{ with } [P_X, P_Y] = 0 \text{ and } \|P_X\| \le 1. \quad (1)$$

We denote the number of spins $N \equiv |V|$. The term P_X is supported on the subset X of the spins; i.e., it acts trivially on the complement $\bar{X} = V - X$ of X. The Hamiltonian is local in the sense that $P_X = 0$ whenever X has radius larger than some constant w. Because we are only interested in the ground state and scaling of the energy gap, we can assume without loss of generality that each P_X is a projector. We also assume that H is frustration free, meaning that the ground states minimize the energy of each term of the Hamiltonian separately, i.e., $P_X|\psi_0\rangle = |\psi_0\rangle$. Then, the ground space $\mathcal{C}$ is the image of the code projector $P = \prod_X P_X$ (henceforth, the $P_X = 0$ are not included in such products).

This family of lattice models, called local commuting projector code (LCPC) includes most known models of topological order including quantum doubles [2], Levin-Wen [7], and Turaev-Viro [8] models. It has been proved that LCPCs have a stable spectrum [9–12] if they obey the following local topological order condition.

Definition (Local topological order).—For any topologically trivial region A, let $P_A = \prod_{X: X\cap A \neq \varnothing} P_X$ be the product of projectors that intersect region A. For a system with local topological order, $\rho^A \equiv \mathrm{Tr}_{\bar{A}} P$ has the same kernel as

$\rho_A^{\rm loc} \equiv {\rm Tr}_{\bar{A}} P_A$ and moreover ${\rm Tr}_{\bar{A}} \psi \propto \rho^A$ for any ground state $|\psi\rangle$.

Our main result is that any system with local topological order has only a constant energy barrier between ground states. This result can be understood intuitively when the low-energy excitations of the system are localized deconfined anyons. In that case, it is possible to modify the ground state by creating an anyon pair, dragging one of them along a topologically nontrivial loop and annihilating it with its partner. This process clearly requires only a constant amount of energy—the mass gap—because the anyons are deconfined. The obstacle in formalizing this heuristic picture is to prove that the low-energy excitations are indeed localized deconfined anyons. Toy models of topological order [2,7,8] usually take such an anyon model as a starting point to construct a local Hamiltonian. Here, we take the opposite path: our starting point is any Hamiltonian with local topological order and we want to characterize its low-energy excitations. This is an active area of research [13]. It is unclear whether excitations are always pointlike in these models and, as noted by Haah and Preskill [14], whether creating and moving them can always be realized by a sequence of local unitary transformations. Thus, our result is a step toward a general understanding of low-energy excitations in models with local topological order, which should be of independent interest. Indeed, our proof will essentially confirm the above heuristic picture, but it circumvents the delicate unitarity issue.

Background.—Characterizing the thermal stability of a memory requires detailed knowledge of its thermalization process. Because we seek to address a broad class of systems, our analysis cannot be model specific. We thus retain only two essential features common to all thermalization processes: (i) the bath interacts locally with the system, and (ii) high-energy states are penalized. As we now explain, we can combine these features to obtain a sufficient condition for thermal stability.

We will say that a memory with Hamiltonian Eq. (1) has an energy barrier at most Δ if there exist a ground state ψ_0 and a sequence of $T \in {\rm poly}(N)$ completely positive trace preserving maps $\mathcal{E}_k$, each acting locally on the system and a finite-dimensional ancilla A, such that (i) starting from ψ_0, the sequence returns the system to the ground space, (ii) in a state that differs from the initial state ψ_0, and (iii) the energy of any intermediate state is at most Δ above the ground state energy E_0. More formally, these conditions are

$$ {\rm Tr}[P \cdot \mathcal{E}_T \ldots \mathcal{E}_2 \mathcal{E}_1(\psi_0 \otimes \rho_A)] \geq \tfrac{2}{3}, \quad (2a)$$

$$ {\rm Tr}[\psi_0 \cdot \mathcal{E}_T \ldots \mathcal{E}_2 \mathcal{E}_1(\psi_0 \otimes \rho_A)] \leq \tfrac{1}{3}, \quad (2b)$$

$$ \max_{k \in 1,2,\ldots,T} {\rm Tr}[H \cdot \mathcal{E}_k \ldots \mathcal{E}_2 \mathcal{E}_1(\psi_0 \otimes \rho_A)] - E_0 = \Delta, \quad (2c)$$

where P is the code projector and the factors $\frac{2}{3}$ and $\frac{1}{3}$ are arbitrarily chosen constants. The additional ancillary system, initially in state ρ_A, is used to model some finite non-Markovian effects of the bath, so that each map $\mathcal{E}_k$ has complete access to it. The energy barrier of a memory is taken to be the smallest value of Δ over all such sequences of maps. If a memory has a macroscopic energy barrier $\Delta \geq N^\alpha$ for some constant $\alpha > 0$, then any short sequence of local transformations that returns the system to an altered ground state must visit a high energy state and is therefore thermally stable. Our main result is obtained by exhibiting a sequence of maps $\mathcal{E}_k$ with an energy barrier Δ that is a constant, independent of the system size N for any LCPC. In addition, we show that the length T of this sequence is proportional to the linear size of the lattice when the system has local topological order.

We call a logical operator an operator L that maps the ground space to itself, i.e., $[L, P] = 0$. In particular, we are interested in logical operators that act nontrivially on the code space, i.e., $LP \neq P$, as they can alter the encoded information. In a series of papers [14–17], it was shown that any 2D LCPCs admit at least one nontrivial logical operator supported only on a 1D (constant width) strip of the lattice. However, it was unclear how to apply it through a sequence of local transformations.

An important subclass of LCPCs is stabilizer codes [18], for which $P_X = \frac{1}{2}(I + S_X)$ where S_X is a tensor product of Pauli matrices $\sigma_{0,1,2,3}$ (with σ_0 being the identity I). Because of this particular structure, the nontrivial string-like logical operator described above is also a tensor product of Pauli matrices, $L = \bigotimes_{k=1}^{\ell} \sigma_{j_k}^k$ where k labels the ℓ sites along the strip in some natural way, from left to right, say. Then, applying the error sequence $\{\sigma_{j_k}^k\}$ will build up to the operator L and will only visit intermediate states with a constant energy above the ground state. Indeed, at an intermediate stage n, $0 < n < \ell$, only a segment $\bigotimes_{k=1}^{n} \sigma_{j_k}^k$ of the logical operator L has been applied. This segment commutes with all terms P_X except the ones within distance $\mathcal{O}(w)$ from site n, so that only these terms contribute to the energy: the excitations are pointlike objects located in the vicinity on the end of the string segment, and so the memory is unstable [15,17,19,20]. This simple argument fails for more general LCPCs because logical operators do not have a tensor product structure.

Noise model.—In this Section, we present an error sequence $\{\mathcal{E}_k\}$ that achieves a constant energy barrier. To simplify the presentation, we coarse-grain the lattice—i.e., we partition the lattice into balls of radius w and view each ball as a site occupied by a single D-level spin with $D = d^{\mathcal{O}(w^2)}$—so that we can assume without lost of generality that (i) Λ is a regular $\ell \times \ell$ square lattice; (ii) the nonzero terms P_X in Eq. (1) act only on 2×2 cells; and (iii) there exists a nontrivial logical operator supported on a single line $\mathcal{L}$ of the lattice. Projectors whose support intersects $\mathcal{L}$ define the strip projector $P_{\mathcal{L}} = \prod_{X \cap \mathcal{L} \neq \emptyset} P_X$ supported on the extended strip $\mathcal{L}'$; see Fig. 1. Similarly, projectors whose support intersects sites $k-1$ and k on $\mathcal{L}$ define local constraints $P_{k-1,k} = \prod_{X \cap \{k-1,k\} \neq \emptyset} P_X$.

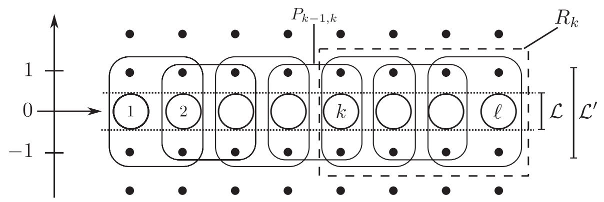

FIG. 1. The strip $\mathcal{L}$ contains ℓ sites (large circles) whose Cartesian coordinates are $\{(k, 0) 1 \leq k \leq \ell\}$. Local constraints $P_{k-1,k}$ act on nearest-neighbor sites $k-1$ and k and on particles in the extended strip $\mathcal{L}' = \{(i, j) | j| \leq 1\}$.

The sequential noise model is a sequence of individual iterations for every site $k \in \mathcal{L}$. Each iteration consists of several trials. Trial m of iteration k corresponds to (i) applying a trial unitary transformation $U_k^{(m)}$ on site k, chosen at random from the Haar measure, and (ii) measuring the local constraint $P_{k-1,k}$. Trials are repeated until a successful trial in which the $+1$ outcome of $P_{k-1,k}$ is obtained and the next iteration begins. Given the state on the strip, a unitary transformation is eligible if it leads to a successful trial with nonzero probability. The initial iteration $k = 1$ differs because the constraint is not measured. Physically, the whole procedure corresponds to creating a random excitation at iteration 1, and moving it along the strip across to the opposite edge by subsequent iterations.

The sequential noise model only creates intermediate states of constant energy. The reason is the same as for stabilizer codes: the excitations are pointlike objects. Indeed, during iteration k, the state is almost everywhere indistinguishable from a ground state because it obeys all constraints $P_{i-1,i}$, except $P_{k-1,k}$ and $P_{k,k+1}$ because only those potentially do not commute with $U_k^{(m)}$. Furthermore, a failed trial during iteration k does not affect the outcome of previous iterations because local constraints commute. Next, we prove that each trial has a constant success probability and that the sequential noise model has a nontrivial effect on the ground space.

Expected number of trials.—The sequential model would run into a dead end, an iteration requiring an infinite number of trials, if the state of the strip admits no eligible unitary transformation at the kth iteration [21]. Such a dead end occurs in the Ising-like toric code introduced in Ref. [9], where the plaquette operators B_p of the toric code are replaced by an Ising-like interaction $B_p B_q$ whose symmetry is broken by a single defect plaquette B_{p^*} to recover the toric code ground space. The sequential model could start preparing the $B_p = -1$ sector and reach a dead end when it encounters the $B_{p^*} = +1$ constraint. However, that code does not have local topological order, and its spectrum is unstable (B_{p^*} is a local order parameter). We now show that dead ends do not occur with local topological order.

Proposition 1.—Local topological order implies that, at any iteration k, there exists an eligible unitary transformation U_k.

Proof.—We will prove the contrapositive. Let k be the first iteration where no U_k is eligible and let ψ be the state during this iteration. We have

$$P_{k-1,k} U_k |\psi\rangle = 0 \quad \forall\ U_k. \tag{3}$$

Mathematically, applying a Haar-random unitary operator is on average equivalent to applying the maximally depolarizing channel $\mathcal{D}_k$,

$$\mathcal{D}_k[\psi] \equiv \mathrm{Tr}_k[\psi] \otimes I_k/D = \int U_k |\psi\rangle\langle\psi| U_k^\dagger dU_k. \tag{4}$$

Thus, the average of Eq. (3) over the Haar measure, is

$$P_{k-1,k}(\mathrm{Tr}_k[\psi] \otimes I_k/D) = 0. \tag{5}$$

Tracing out the region $R_k = \{(i, j): i \geq k, |j| \leq 1\} \subset \mathcal{L}'$ of the extended strip located at the right of site k (cf. Fig. 1), Eq. (5) yields

$$\mathrm{Tr}_k[P_{k-1,k}] \mathrm{Tr}_{R_k}[\psi] = 0. \tag{6}$$

Thus, there exists a state $|\xi\rangle$ in the support of $\mathrm{Tr}_{R_k}[\psi]$ that is in the image of $P_{i-1,i}$ for $i < k$ but also is in the kernel of $\mathrm{Tr}_k[P_{k-1,k}]$. This entails violation of local topological order on site $k-2$ since $\mathrm{Tr}_{\overline{k-2}}[\xi]$ is in the kernel of $\rho_{k-2} = \mathrm{Tr}_{\overline{k-2}} P$ but in the image of $\rho_{k-2}^{\mathrm{loc}} = \mathrm{Tr}_{\overline{k-2}}[P_{k-3,k-2} \times P_{k-2,k-1}]$. ■

Proposition 2.—When the system has local topological order, the expected number of trials A_k at iteration k is finite and independent of the system size.

Proof.—We introduce two maps,

$$\mathcal{P}_{k-1,k}[\rho] = P_{k-1,k} \rho P_{k-1,k}, \tag{7}$$

$$\mathcal{Q}_{k-1,k}[\rho] = (I - P_{k-1,k}) \rho (I - P_{k-1,k}), \tag{8}$$

which represent a successful and failed measurement of the local constraint $P_{k-1,k}$. In a failed trial, the map $\mathcal{Q}_{k-1,k}$ is always immediately preceded and followed by a depolarization of site k. This sequence can be rewritten in an equivalent form

$$\mathcal{D}_k \mathcal{Q}_{k-1,k} \mathcal{D}_k = \mathcal{B}_{k-1} \otimes \mathcal{D}_k, \tag{9}$$

which defines a biasing map $\mathcal{B}_{k-1}$. This map is not trace preserving because the trace of its unnormalized output state is the average probability of a failed trial.

The sequence of m failed trials followed by a successful trial produces the map

$$\mathcal{P}_{k-1,k} \mathcal{D}_k (\mathcal{Q}_{k-1,k} \mathcal{D}_k)^m = \mathcal{P}_{k-1,k} (\mathcal{B}_{k-1}^m \otimes \mathcal{D}_k), \tag{10}$$

where we have used Eq. (9) and $\mathcal{D}_k^2 = \mathcal{D}_k$. Thus, given a state ψ, the average probability $p_k^{(m)}(\psi)$ of a success after m failures is $p_k^{(m)}(\psi) = \mathrm{Tr}[\mathcal{P}_{k-1,k}(\mathcal{B}_{k-1}^m \otimes \mathcal{D}_k)[\psi]]$. Therefore, the expected number of trials

$$A_k(\psi) = \sum_{m=1}^{\infty}(m+1)\mathrm{Tr}[\mathcal{P}_{k-1,k}(\mathcal{B}_{k-1}^m \otimes \mathcal{D}_k)[\psi]] \quad (11)$$

$$= \mathrm{Tr}[\mathcal{P}_{k-1,k}((\mathcal{I}_{k-1} - \mathcal{B}_{k-1})^{-2} \otimes \mathcal{D}_k)[\psi]] \quad (12)$$

is bounded by the norm of the superoperator inside the trace and thus only depends on the microscopic details of H, not on system size. Note that the geometric sum of Eq. (12) converges because $\mathcal{B}_{k-1}$ cannot have $+1$ eigenvectors in the ground space for a local topological ordered system, because those would contradict proposition 1 [22]. ■

Nontrivial average effect.—We now prove that the sequential noise model corrupts the encoded information. The effect of a single iteration k averaged over all possible trials amounts to the map

$$\mathcal{E}_k = \sum_{m=0}^{\infty} \mathcal{P}_{k-1,k}(\mathcal{B}_{k-1}^m \otimes \mathcal{D}_k) \quad (13)$$

$$= \mathcal{P}_{k-1,k}[(\mathcal{I} - \mathcal{B}_{k-1})^{-1} \otimes \mathcal{D}_k], \quad (14)$$

and the average total effect of the sequential noise model $\mathcal{E} \equiv \prod_{k=1}^{\ell} \mathcal{E}_k$ is

$$\mathcal{E} = \prod_{k=2}^{\ell} \mathcal{P}_{k-1,k}[(\mathcal{I} - \mathcal{B}_{k-1})^{-1} \otimes \mathcal{D}_k]\mathcal{D}_1. \quad (15)$$

Terms with nonoverlapping support trivially commute. We thus move all depolarizing channels to act first, which globally depolarizes the strip. To move the biasing operators past the projectors, it suffices to prove that $C \equiv [\mathcal{B}_k, \mathcal{P}_{k-1,k}]\mathcal{D}_{k+1}$ is zero. Because of their nonoverlapping supports, $\mathcal{D}_{k+1}$ commutes with $\mathcal{P}_{k-1,k}$ and $C=[\mathcal{B}_k\mathcal{D}_{k+1}, \mathcal{P}_{k-1,k}]=[\mathcal{D}_{k+1}\mathcal{Q}_{k,k+1}\mathcal{D}_{k+1}, \mathcal{P}_{k-1,k}]=0$ because $\mathcal{Q}_{k,k+1}$ and $\mathcal{P}_{k-1,k}$ commute. Hence, the terms of Eq. (15) can be reordered into

$$\mathcal{E} = \prod_{k=2}^{\ell} \mathcal{P}_{k-1,k} \prod_{k=1}^{\ell}(\mathcal{I} - \mathcal{B}_k)^{-1} \prod_{k=1}^{\ell} \mathcal{D}_k. \quad (16)$$

Thus, the average effect of the sequential noise model is equivalent to three consecutive transformations. First, all particles on the strip are removed and replaced by particles in random states. At this point, it is clear that the memory has been irreversibly corrupted. Then, an arbitrary transformation is applied on the strip before returning the system to its ground space. Hence, the error sequence $\{\mathcal{E}_k\}_{k=1}^{\ell}$ satisfies conditions from Eq. (2).

Discussion.—It is known that 2D LCPCs have a unique Gibbs state [23], so that they cannot store information in thermal equilibrium. Thus, the question of self-correction is fundamentally about the thermalization time. The noise process we presented corrupts the memory after a time that grows proportionally to the system size, which can be interpreted as a (macroscopic) upper bound to the storage time. However, there are good reasons to believe that the actual storage time is in fact independent of the system size. At nonzero temperatures we expect a finite density of defects, so the noise process we described could be happening in parallel all over the lattice. As pairs of defects meet, they can fuse to the vacuum with some probability to create longer error strings. The memory time is then related to the percolation of these error chains, which should be independent of the system size.

Our result does not completely close the door to the existence of a robust quantum memory in 2D. First, local topological order is a sufficient, but perhaps not necessary condition for spectral stability. Second, we have restricted the form of the Hamiltonian. In realistic physical systems, the terms P_X need not commute, the ground space can be frustrated, and interaction can decay algebraically with the distance between sites. Third, a macroscopic energy barrier is one mechanism that leads to thermal stability, but there may exist other mechanisms. In particular, a system in contact with a heat bath tends to minimize its free energy $F = E - TS$. Thus, we could imagine a system with a large entropy barrier: among all possible local noise sequences, only a vanishingly small fraction will induce a change of sector, while the overwhelming majority lead to dead ends as described above. Such topological spin-glasses [24] could offer an enhanced quantum memory lifetime. This proposal is distinct from existing studies showing that disorder induces an exponential localization of anyons [25–28], as those only address zero-temperature storage.

Conclusion.—Our main result hints at a general trade-off in 2D between a quantum memory's ability to suppress thermal and quantum fluctuations. Recent discoveries [29–31] indicate that this trade-off is not necessary in 3D. Our result extends prior findings [15,17] derived for stabilizer codes to a broader, widely studied class of models that includes quantum doubles [2], Levin-Wen [7], and Turaev-Viro [8] models. It also generalizes straightforwardly to higher dimensions for systems that have quasi-one-dimensional nontrivial logical operators.

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