

Comment on “Some non-conventional ideas about algorithmic complexity”

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We comment on a recent paper by D’Abramo [Chaos, Solitons & Fractals, 25 (2005) 29], focusing on the author’s statement that an algorithm can produce a list of strings containing at least one string whose algorithmic complexity is greater than that of the entire list. We show that this statement, although perplexing, is not as paradoxical as it seems when the definition of algorithmic complexity is applied correctly.

D’Abramo has advanced in a recent paper [1] a number of ideas related to the notion of algorithmic complexity (also called Kolmogorov or Chaitin complexity [2]), and what seems to be a paradox related to the definition of this quantity. In our opinion, D’Abramo’s paper may leave the impression that the very definition of algorithmic complexity is contradictory or inconsistent as a result of this apparent paradox. This, in fact, is not the case, as we would like to show now.

To start, let us recall the central point of D’Abramo’s considerations [1, p. 29]. Consider a program p that outputs all binary strings of length less than or equal to N bits in lexicographic order, that is, that outputs the sequence

$$\ell = 0 \ 1 \ 00 \ 01 \ 10 \ 11 \ 000 \ 001 \ 010 \ 100 \ 110 \ 101 \ 011 \ 111 \dots \underbrace{11\dots 1}_{N \text{ bits}}.$$

The program p must obviously contain the number N (to know when to halt), plus some other bits of instructions for generating the strings and printing them. As a result, p is of length $\lceil \log_2 N \rceil + k$ bits when written in binary notation, where k is some N -independent constant that accounts for the overhead instructions. From this reasoning, we conclude that the algorithmic complexity of the list ℓ is at most $\lceil \log_2 N \rceil + k$ bits. (Note that the number N could be describable with less than $\log_2 N$ bits.)

Now comes the apparent paradox. The number of binary strings contained in ℓ is $2^{N+1} - 2$, while the total number of programs having length at most $\lceil \log_2 N \rceil + k$ is $2^{\lceil \log_2 N \rceil + k + 1} - 2$. For N sufficiently large, we have

$$2^{N+1} - 2 \gg 2^{\lceil \log_2 N \rceil + k + 1} - 2,$$

which means that ℓ contains a lot more binary strings than what can be produced by all programs of length at most $\lceil \log_2 N \rceil + k$. Consequently, there must be a least one string s in ℓ having a complexity greater than

$\lceil \log_2 N \rceil + k$, since there must be at least one string in ℓ which is not produced by a program of length at most $\lceil \log_2 N \rceil + k$. This is the contradiction or “paradox” that D’Abramo alludes to: ℓ contains a string s of complexity greater than the complexity of ℓ itself, that is, greater than the length of the minimal program that generates it. In D’Abramo’s words [1, p. 30]: “Suddenly a paradox appears: an algorithm [...] is able to write a list of strings which contains at least one that is more complex than [...] the algorithm itself.”

This last statement is quite perplexing. If a program could indeed output something more complex than itself, then the very definition of algorithmic complexity would be inconsistent. Fortunately, this is not the case, and there is a precision to be made here. The fact is that D’Abramo’s program does *not* output the string s that has a complexity greater than that of ℓ —it outputs much more. Therefore, it is misleading to allude to the complexity of s in the context of the program p , and then compare it with the complexity of ℓ ; p generates ℓ not s .

To define the complexity of s , we need to find a minimal program that produces it. Following D’Abramo, one can attempt to modify the program p so that it first produces ℓ , and then selects within it a given string s as the final output [1, p. 31]: “[the program] can then stop the counter whenever it wants and print the last enumerated number. The counter could be provided with a sort of counting completeness indicator, which would give the percentage of the whole count reached till that moment.” In order to work, this new program needs to contain N as before, but also requires the address of the string s , which in this case corresponds to the string s itself. In other words, to tell the “counter” when to stop and print the final output requires a number of bits equal to the shortest description of the desired output s . Hence, the new program is at least as long as the shortest description of s , and so what appeared at first to be a paradox now has a simple explanation.

The same conclusion can be reached from a different perspective by noting the nonadditivity of the algorithmic complexity, that is, the fact that the complexity $K(x, y)$ of the union of two strings x and y is not necessarily equal to the sum of their individual complexities $K(x)$, $K(y)$. Rather, the joint complexity satisfies the inequality $K(x, y) \leq K(x) + K(y|x)$, where $K(x|y)$ is the length of the shortest program that generates y given x as an input [2]. Applying this inequality recursively to the entire list shown in (1), it is not surprising that the complexity of the list can be smaller than that of some of its components.

In closing this comment, it may be of interest to point out that ℓ is equivalent to the Champernowne constant [3], which is known to be random according to the Shannon definition, that is, normal in the language of number theory, and yet has a low algorithmic complexity [2].

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- [3] D. G. Champernowne, *J. London Math. Soc.* 8 (1933) 254.