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## **A Rough Guide to Quantum Chaos**

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This tutorial offers some insight into the question “*What is quantum chaos and why is it interesting?*”. Its main purpose is to present some signatures of chaos in the quantum world. This is *not* a technical reference, it contains but a few simple equations and no explicit references; rather, the main body of this manuscript is followed by a reading guide. Some of the mathematical tools used in the field are so cumbersome they often obscure the physical relevance of the problem under investigation. However, after having consulted this tutorial, the technical literature should appear less mysterious and, we hope, one should have a better intuition of what is interesting, and what is superficial!

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### **Why quantum chaos?**

Let us set the record straight: there is no such thing as quantum chaos. The term “quantum chaos” is a shorthand for the study of quantized systems whose classical analog exhibits chaotic features. This raises two obvious questions: why does quantum chaos not exist and, since that is the case, why is the study of quantized chaotic systems of interest? In what follows, we will try to clarify these two issues.

In classical dynamics, the standard definition of chaos is formulated in terms of hypersensitivity to initial conditions. Two arbitrarily close points in phase space will grow apart from each other at an exponential rate. This rate is fixed by the largest Lyapunov exponent which characterizes the dynamics of the system. As soon as one of the Lyapunov exponent is positive, some trajectories in phase space

will diverge exponentially in time so we say that the system is chaotic. For Hamiltonian systems, the sum of the Lyapunov exponents must be unit since the dynamics is conservative, i.e. it preserves volume in phase space. Therefore, if some trajectories diverge exponentially under some Hamiltonian there must be some trajectories which converge exponentially. Only Hamiltonians which are nonlinear functions of position and momentum can exhibit chaotic features. In the absence of such features the system is said to be regular or integrable.

Quantum mechanical systems are not represented by points in phase space but, rather, by vectors in Hilbert spaces. The time evolution of these vectors is governed by the linear Schrödinger equation. This is the crucial distinction which is responsible for the absence of chaos in quantized systems. For classical chaotic systems, the distance between two points in position and momentum space can grow exponentially in time because the dynamics depends in a nonlinear fashion on position and momentum themselves. On the other hand, the dynamics of a state vector does *not* depend on the vector itself: the quantum equation of motion is linear. Since nonlinearity was a crucial ingredient for exponential divergence, quantum systems cannot exhibit hypersensitivity to initial conditions. Moreover, it follows from the Hermiticity of the Hamiltonian, i.e. the fact that the energy of a system must be a real number, that the evolution operator is unitary. While linearity of the dynamics constrained the growth in time of the distance between two vectors to be subexponential, unitarity has a more drastic consequence: the distance between any two vectors is a constant of motion. This is more than needed to rule out the possibility of exponential divergence between initially close quantum states.

The linearity of Schrödinger's equation ensures that there is no such thing as chaotic quantum dynamics defined in terms of hypersensitivity to initial conditions; so what's the interest of quantum chaos? Well the simple fact that this definition fails has a certain interest. Since all systems are fundamentally quantum, how can some of them, such as the solar system, exhibit hypersensitivity to initial conditions? Is it that the predictions of quantum mechanics are wrong? Hence, a large area of research in the field of quantum chaos consist in finding other criteria which characterize classical chaos and can be adapted to the quantum world. An example of such criteria is the well known butterfly effect: hypersensitivity to perturbation. Instead of looking at how a small difference in initial conditions evolve in time, one can study the different evolutions of two initially identical systems whose dynamics are governed by two slightly different Hamiltonians. If the dynamics of the system is chaotic, this difference should grow exponentially before saturating at a certain maximal value while for regular systems, this growth should follow some power law.

If the alarming discrepancy between classical chaos and the prediction of quantum mechanics was only caused by an “incorrect definition of chaos”, the interest in quantum chaos would be rather superficial, a pedagogical curiosity at most, but would certainly not deserve the attention it is presently being devoted. Thus, it should not come as a surprise to the reader to learn that chaos creates a second equally alarming conflict between the predictions of quantum mechanics and those of classical dynamics: the Ehrenfest correspondence principle breaks down after a short time for chaotic systems. Recall that Ehrenfest’s principle stipulates that in the limit of large quantum numbers, the quantum average of the position and momentum operators should reproduce a classical paths in phase space. It was shown by Ehrenfest, using a generic system, that the time required for the difference between the two predictions to be noticeable was sufficiently long for the experimentalists to sleep quietly! What escaped him at the time was that those predictions differ on much shorter time scales when the system under investigation is chaotic. Therefore, it is of crucial importance to understand under what conditions the correspondence principle can be reestablished.

Last but not least, an important byproduct of quantum chaos is our better understanding of many body systems. This connection is established through the great statistical agreement between spectral properties of chaotic quantum systems and those of random matrices. Historically, random matrix theory was introduced in physics to study complex many body Hamiltonians. The classical analog of these systems are generally not integrable due to their small number of conserved quantities compared to the number of degree of freedom they possess. It was later realized that the spectral properties of random matrices could also reproduce those of quantized chaotic systems involving only a few degree of freedom. In the technical jargon, the complex dynamics caused by a few degrees of freedom is called deterministic chaos; when it is caused by many body interaction, we name it microscopic chaos. Here, we mainly concentrate on the former, but would like to add this brief comment about the latter. Perhaps the most interesting aspect of microscopic chaos is its connection with thermodynamics, more specifically to our understanding of the second law. If the increase of entropy is to be thought of as a fundamental feature of our universe, better make it quantum mechanical. Therefore, a deep understanding of quantum microscopic chaos should yield a detailed explanation of the second law; unfortunately, this connection will not be discussed in what follows.

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## Quantum-classical correspondence

Bohr's correspondence principle is one of quantum mechanics' cornerstone. It states that in certain classical limits, quantum theory should reproduce the predictions of classical theory with vanishing errors. In particular, for those objects which are known to be in excellent agreement with classical mechanics — chairs, planets, etc. — quantum effects should be negligible. In fact, this can be established formally under some assumptions. Consider a simple Hamiltonian  $H = P^2/2m + V(X)$  where  $P$  and  $X$  the position and momentum operators. Now, assume that the initial quantum state  $|\psi(0)\rangle$  is localized around the phase point  $(x(0), p(0))$  and has a small spread, as small as Heisenberg's uncertainty allows. Then, the quantum expectation values of  $x(t) = \langle \psi(t) | X | \psi(t) \rangle$  and  $p(t) = \langle \psi(t) | P | \psi(t) \rangle$  are governed by equations which are almost identical to Newton's equation. The only difference is that the force  $F(x)$  is replaced by its quantum expectation value  $\langle \psi(t) | F(X) | \psi(t) \rangle$ . Nevertheless, under our working assumptions, these terms are almost identical: the first correction is proportional to the spread in position of the wave function times the second derivative of the force. Higher order terms may also become significant because they involve increasing order moments of  $X$  and derivatives of  $V$ . Of course, these higher derivatives can only contribute when  $V$  involves super-quadratic powers of  $X$ . In a regime where the variations of the potential are on a much larger scale than the spread of the wave function *and* this latter is sufficiently large, the principle, known as Ehrenfest's correspondence, should hold.

Given a potential  $V(x)$ , it is crucial to determine for how long a localized wave function will remain localized, on a scale set by the typical variation of the potential. Indeed, by virtue of Heisenberg's principle, a finite spread in position implies a finite spread in momentum. this position uncertainty will increase the spread in momentum through dynamical effects — the particle is traveling at “different speeds” so it gets delocalized — but this can take a long time. This *break-time* can be estimated and, for generic Hamiltonians, it is proportional to some characteristic power (of order unity) of the typical action of the system (in units of  $\hbar$ ). Since the quantum unit of action is so small ( $10^{-34} J \cdot s$ ) compared to the action involved for macroscopic objects, the Ehrenfest break-time is usually astronomical. For example, the typical pendulum found in clocks (100 cm arm, 2 sec. oscillation and 100 g mass) has a typical action  $S/\hbar = 2 \times 10^{33}$ .

The major omission of the previous demonstration, which had many historical consequences, was to evaluate the behavior of the break-time only for regular dynamic. If one considers chaotic Hamiltonians, the Ehrenfest break-time does not

follow some power law with respect to the action but, for a variety of models, a logarithmic law. Perhaps the most drastic consequences of this law can be illustrated with the help of Hyperion, a moon of Saturn widely studied for its chaotic motion, for which the estimated break-time is of the order of ten years. After this period, the classical and quantum predictions are expected to diverge on a scale of many kilometers!

### Quantum localization

To fully grasp the origin of this discrepancy, we shall illustrate it on a simple chaotic model: the kicked rotator. The rotator has two variables, its momentum and an angle, which we shall refer to as  $p$  and  $\theta$ . The phase space is a cylinder:  $p$  can take all real values while  $\theta$  is restricted to  $[0, 2\pi)$  with periodic boundary conditions. The dynamics of this system is specified by its stroboscopic effect:  $\theta_{t+1} = (\theta_t + p_t) \bmod 2\pi$  and  $p_{t+1} = p_t + \lambda \sin \theta_{t+1}$ . In words, the angle rotates around the cylinder at a velocity prescribed by  $p$  while the variation in time of the momentum is caused by a potential which has a cosine profile. For high values of  $\lambda$ , the angle may get “wrapped around” the cylinder many times; to a good approximation for the classical analysis, we may assume that the angle is therefore memory-less. As a consequence, the momentum increment ( $\lambda \sin \theta_{t+1}$ ) is well modeled by a random variable distributed symmetrically over  $[-\lambda, \lambda]$ . Thus the average value of the momentum is equal to its initial value  $p_0$ . On the other hand, and this is crucial, its variance (width of the distribution) grows linearly in time, unboundedly.

It is quite simple to show that this classical map is chaotic. Indeed, define  $\delta p_t$  and  $\delta \theta_t$ , the difference between two phase points at time  $t$ . The recursion relation relating this difference at two successive times can be expressed as a  $2 \times 2$  matrix multiplication. Hence, to compute this distance at time  $t$  given it at time 0 requires the multiplication of  $t$  such matrices. Using a theorem by Furtenberg, it can be shown that the resulting matrix has an exponentially large eigenvalue (and an exponentially small one as imposed by the conservation of phase space volume of Hamiltonian systems): exponential divergences of paths in phase space.

Let’s turn now to the quantum case. The first part of the analysis establishes a relation between the kick rotator and Anderson’s model. In a simple version of this model, a particle can hop from site to site in a one-dimensional chain. Each site of the chain has a random potential and hopping from one site to the neighboring one costs a fixed amount of energy. An eigenstate of this Hamiltonian  $H|\phi\rangle = E|\phi\rangle$  has amplitude  $\phi_m$  on site  $m$ . The value of  $\phi_{m+1}$  can be computed recursively, it is related to  $\phi_{m-1}$  and  $\phi_m$  via a  $2 \times 2$  matrix. Once again, we use Furtenberg’s

theorem, but this time it is the exponentially small value which is of interest to us. Indeed, the growing exponential does not represent a physical solution because such a wave function would not be normalizable. We are thus forced to conclude that all eigenstates of this Hamiltonian are localized around some site: their magnitude decrease exponentially with the distance from their center.

The site in Anderson's model played the role of the variable  $p$  in a discretized version of the kicked rotator. The conclusion drawn from this analogy is that the eigenstates of the kicked rotator are exponentially localized in  $p$  space, with some characteristic width  $\ell$ . Assume that the initial state of the system is  $\psi_0$  and it has some finite width  $\sigma_0$  in  $p$  space. Clearly,  $\psi_0$  can only overlap considerably with a finite number of eigenstates of the Hamiltonian. All eigenstates well outside this initial spread  $\sigma_0$  can only have an exponentially vanishing overlap with  $\psi_0$  since they also are localized. Furthermore, it is a straightforward consequence of Schrödinger's equation that the magnitude of the overlap with an eigenstate is constant in time. Thus, the evolution of this wave packet is restricted to a finite range of  $p$ . It can initially spread out like in the classical case but at a critical width — which depends on the value of  $\ell$  and  $\sigma_0$  — the *spreading must cease*. At this time, the evolution of the wave function becomes quasi-periodic; this is a direct consequence of the finite number of eigenvalues involved in the equation of motion.

In this simple model, Anderson's localization — a purely quantum phenomenon — creates an unbounded discrepancy between quantum and classical predictions. While the spread in momentum of the classical rotator increases for ever, its growth for the quantum rotator halts after some critical break-time.

## Decoherence

The exponential stretching of phase space induced by chaotic Hamiltonians challenges the correspondence principle. As argued above, this stretching must be accompanied of an exponential squeezing in order to preserve volume. But a direct consequence of this stretching and squeezing is the violation of the conditions under which Ehrenfest's principle is valid. Exponential squeezing will inevitably bring significant quantum corrections through high order terms. But yet, we do not see quantum features in the macroscopic world surrounding us; this world is very well modeled by classical mechanics.

The weak interaction of a system with its environment is a well known remedy to apparent conflicts between quantum and classical mechanics. Decoherence is the name given to the suppression of quantum effects caused by the existence of *uncontrolled* degrees of freedom which contribute to the system's dynamics.

These degree of freedom can be external (heat bath, cosmic background radiation, air molecules, etc.) or internal (fine and hyperfine structure of an atom, molecular structure of a macroscopic object, etc.); they are generally referred to as *the environment*. Roughly speaking, decoherence selects a set of “preferred states” of the system which, for typical models, coincide with states one would naturally qualify as “classical”, those states which are to a certain degree localized in phase space. The effect of the coupling to an environment is to induce a superselection rule on the system, forbidding it to be in a superposition of preferred states. A system initialized in such a superposition will *decohere* into a classical mixture of the preferred states after a time which depends on the strength of the coupling to the environment. Hence, decoherence provides an explanation to why we do not see cats in weird dead/alive superposition like those which should would emerge from Schrödinger *Gedankenexperiment*.

But can decoherence reestablish the quantum-classical correspondence for chaotic systems? The answer appears to be yes. To understand this, it is helpful to think of the environment as an observer repeatedly measuring the system in the preferred basis. By the very nature of the environment, the outcome of this measurement remains unknown; this is why the wave packet is transformed to a statistical mixture. Hence, before the wave packet becomes exponentially stretched, it is collapsed to a mixture of some roughly local states. Consequently, the squeezing also gets interrupted before high order quantum corrections become significant.

The effect of decoherence can be illustrated with the help of Anderson’s model. As argued above, a wave packet centered at momentum  $p_0$  and with a finite width  $\sigma_0$  can only overlap considerably with a finite number of localized eigenstates of the Hamiltonian. Depending on the decoherence time scale, the system will undergo a certain number of iterations before being measured by the environment. During these iterations, the wave packet will have spread at a linear rate from  $\sigma_0$  to  $\sigma_t$ : recall that it is only after a critical break-time that this linear increase halts. The “outcome” of the environmental measurement can be anyone of the preferred states that is localized within the wave packet’s spread  $\sigma_t$ . After this measurement, the quantum dynamics is reinitialized according to the outcome so dynamical localization never enters the picture.

The previous example clearly illustrates that decoherence can suppress the quantum effects responsible for the divergences between quantum and classical predictions. In fact, it is so efficient at doing so that it can serve as a quantum signature of chaos. Since the outcomes of the environmental measurement are not revealed, our knowledge about the system decreases in time. This can be established quantitatively from the increase of the system’s entropy. It was postulated

by Zurek and Paz that this increase of entropy can testify to the presence of chaos. The initial growth for times of the order of the decoherence time depends mostly on the details of the coupling to the environment. After this initial increase, the system enters a second regime which should mostly depend on the system's Hamiltonian, not on the strength of the environmental interaction. In this regime, Zurek and Paz have argued that the rate of increase of entropy should be fixed by the largest Lyapunov exponent of chaotic systems, until the system reaches equilibrium. For a regular system on the other hand, entropy production in this regime should be logarithmic so it only reaches equilibrium after an exponential time.

### **Liouville correspondence**

The correspondence principle seems to be endangered when the predictions of quantum mechanics are compared with those of Newton's mechanics of a *chaotic* system. A weak coupling with the environment will modify quantum mechanical predictions just enough so it no longer enters into conflict with Newton's equation of motion. Nevertheless, maybe this recovery was not desirable since Newton's mechanics is an idealization describing the motion of *points* in phase space. A more reasonable requirement would be to recover the predictions of Liouville's mechanics, the dynamics of *probability distributions* on phase space.

There are two reasons to believe that Liouville's equation should yield predictions which are in better agreement with Schrödinger's equation. First, no quantum state can represent a point in phase space, it is prohibited by Heisenberg's principle. All quantum states must have non-zero spread in phase space. Therefore, probability distributions over phase space are much more suited to play the role of quantum states in classical mechanics. To represent a quantum state in phase space, one must use the Wigner representation. A Wigner function behaves just like a probability distribution when it comes time to compute averages but it is *not* a probability distribution; it can take negative values. Hence, it is generally impossible to match the initial conditions perfectly with a classical distribution, but one can hope to reproduce most low order moments to a good accuracy.

Second, the translation of Schrödinger's equation in terms of Wigner functions yields an equation quite similar to Liouville's classical equation. In fact, the only distinction is that the classical Poisson brackets are replaced by Moyal brackets. This latter can be expressed as a Poisson bracket plus some correction terms. While the analysis of these corrections is highly complex, numerical results are encouraging. Inevitably, the results obtained from the quantum and classical equation start diverging after some critical break-time which is roughly equal to Ehrenfest's break-time. In the case where quantum mechanics was compared with point me-

chanics, the divergence was only limited by the system size. However, when compared to distributions on phase space, the divergence saturates much earlier, and on a scale which is independent of the system size. Indeed, numerical results suggest that the disparity typically saturates at a *microscopic* scale, thus saving the precious quantum-classical correspondence. In short, the agreement between the quantum and the classical are improved when the classical standards are set by phase space distributions because they can match the initial conditions to a higher accuracy and are governed by an equation which shows more resemblance to the quantum equation of motion.

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### The quantum butterfly effect

Models of atmospheric dynamic are so complex, it is often said that the flap of a butterfly's wings in Brazil can set off a tornado in Texas! This examples illustrates the atmosphere's extreme sensitivity to initial conditions. The two alternatives of the butterfly — to flap or not to flap its wings — will generate two slightly different initial states which can have huge repercussions on the future (like the creation of a tornado). After its initial intervention, the butterfly does not enter the picture; only the trace of its action on the system remains.

As we now know, this picture has to be modified to enter the quantum world because the “magnitude” of the discrepancy are constant of motion. A possible modification is to keep the butterfly into the picture as a constant disturbance; its options are now to flap its wings repeatedly or not to flap them at all. Hence, the perturbation is made *dynamical*, it is not a property of the system's initial conditions but of its time evolution. The butterfly's action modifies the Hamiltonian of the system and the effects of this modification can serve as a signature of chaos: hypersensitivity to perturbation.

### Perturbation theory

The standard way to deal with a small variations  $\delta V$  of a Hamiltonian  $H$  in the quantum formalism is perturbation theory. The usual assumption is that for sufficiently small  $\delta$ , the eigenvalues/vectors of  $H_p = H + \delta V$  should be equal to those of  $H$  plus some minor corrections expressed as power series of  $\delta$ . Starting from the solution of the unperturbed Hamiltonian  $H|\phi_j\rangle = E_j|\phi_j\rangle$ , one computes corrections to be brought to arrive at the eigenvectors for the perturbed Hamiltonian  $H_p|\psi_k\rangle = D_k|\psi_k\rangle$ . Hence, this technique assumes a one-to-one correspondence between the eigenstates of  $H$  and those of  $H_p$ ,  $\psi_j \simeq \phi_j$  and  $E_j \simeq D_j$ .

It is a straightforward exercise found in all elementary book on quantum mechanics to compute the coefficient of the power series. They involve increasing powers of the perturbation matrix elements  $(V^n)_{ij} = \langle \phi_i | V^n | \phi_j \rangle$ . As will become clear in the next section, the Hamiltonian having a classical regular limit should be block diagonal. As a consequence, the successive power of  $V$  are expected to remain constant so the high order corrections vanish as  $\delta^n$ . On the other hand, chaotic Hamiltonian do not have this block diagonal structure so high powers of  $V$  reach a magnitude which scales as the system size: perturbation theory fails.

As a consequence of this failure, there is no one-to-one mapping between the eigenstates of  $H$  and  $H_p$ . To illustrate this, one can express the eigenstates of  $H_p$  as a superposition of those of  $H$  as  $|\psi_k\rangle = \sum_j \alpha_j^k |\phi_j\rangle$ . For regular systems, one of the coefficients  $\alpha_j^k$  clearly dominates the sum while for chaotic Hamiltonians, there are typically a few of these coefficients which have non negligible magnitude.

The absence of a one-to-one mapping allows for a good understanding of hypersensitivity to perturbations. Let  $F$  and  $F_p$  be the unitary operations resulting from the application of Hamiltonians  $H$  and  $H_p$  respectively over a unit period of time. Assume that the system is initially in state  $|\Psi\rangle = \sum_j a_j |\phi_j\rangle = \sum_k b_k |\psi_k\rangle$ . After a time  $t$ , the system will be in state  $|\Psi(t)\rangle = F^t |\Psi\rangle$  if the dynamics was not perturbed and in state  $|\Psi_p(t)\rangle = F_p^t |\Psi\rangle$  in the opposite case. To evaluate the repercussions of the perturbation, we compute the overlap of these two states  $|\langle \Psi(t) | \Psi_p(t) \rangle|^2 = \sum_{jk} |a_j^* b_k \langle \phi_j | \psi_k \rangle|^2 + \text{oscillations}$ . When perturbation theory holds, the resulting overlap should thus oscillate around a value which is equal to 1 plus corrections of order  $\delta$ ; the dynamics is not sensitive to small perturbations. On the other hand, when the eigenstates of  $H_p$  appear random relatively to those of  $H$ , the overlap is expected to first rapidly decrease and then oscillate around the value  $1/N$  where  $N$  is the dimension of the system. These predictions are in good agreement with numerical results, even for tiny a perturbation which clearly indicates the existence of hypersensitivity to perturbations for chaotic Hamiltonians.

### Environment as a record

It was argued above that a weak coupling to an environment can help reestablishing the quantum-classical correspondence. Through “measurements”, the environment repeatedly perturbs the evolution of the system. Since the results of the measurements are not revealed, this interaction increases the entropy of the system. This picture can be generalized to other kind of perturbations; the system undergoes a perturbed evolution conditioned on the state of the environment. From a thermodynamics point of view, this entropy increase is not desirable since it decreases the amount of work one can extract from the system. The complexity of the system’s

Hamiltonian also enters into account, chaotic systems generate entropy at a much higher rate — this was even postulated as a signature of chaos — hence decreasing their usefulness for thermodynamic purposes.

To counter this noxious effect, one can try to keep track of the system's state by monitoring the environment. But in thermodynamics, the information one gathers is also noxious. At the end of the day, it must be erased in order for the heat engine to run in a cyclic fashion. But Landauer's principles shows that erasing information has a thermodynamic cost. Therefore, the relevant quantity is the tradeoff between the amount of work gained by learning about the system's state through the environment and the amount of work needed to erase this information.

To study this tradeoff quantitatively, we set a threshold  $H_0$  and ask what's the minimal amount of information  $\Delta I$  which must be extracted from the environment in order to keep the system's entropy below the threshold. If  $H_0 = 0$  we don't tolerate any entropy increase and so we must keep track of the exact state of the environment. Hence,  $\Delta I$  will uniquely depend on the coupling between the system and the environment: it is the information needed to specify a particular realization of the perturbation. The same holds for small values of the threshold. The interesting regime is when  $H_0$  is just below the value of the entropy the system has when the environment is not monitored, i.e. when we are just learning a little bit about the system by interrogating the environment.

In this context, it has been shown numerically that for chaotic systems, the Landauer erasure cost is much larger than the decrease of system's entropy, i.e. of available work. This means that the optimal thermodynamic protocol is simply to let the environment scramble the state of the system without trying to keep track of its effects. For regular systems however, the tradeoff is positive: monitoring the environment allows to extract more work from the system even when the erasure cost is taken into account. This signature of chaos establishes a link between the second law of thermodynamics and hypersensitivity to perturbations.

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## Random matrix theory

The spectrum of a many body system generally has a highly complex form. Nevertheless, given a complete description of the system's Hamiltonian, it is possible to understand this spectrum in all its details; this occurs only when the Hamiltonian contains sufficient symmetries. Without symmetries, the system is non integrable and the details of the spectrum cannot be predicted, only some of its general features can usually be explained.

Given the spectrum of such a system — a heavy nuclei say — one may won-

der if its apparent complexity is due to a fundamental lack of symmetries of the underlying Hamiltonian or is simply a consequence of one's limited mathematical skills. Hence, one would like to determine whether the system possesses some symmetries without having to explicitly determine them.

This is the situation Wigner was facing when he introduced random matrix theory into the picture. His idea was to model the statistical properties of the Hamiltonian with those of a random Hermitian matrix satisfying some symmetries. Symmetries act as constraints on the set of all Hermitian matrices, only a subset of all these matrices can satisfy the required symmetries. To compute statistical properties over this set, we need to define a probability distribution on it. This is achieved by defining a probability distribution over matrix elements  $P(\{M_{ij}\})$  of the matrix  $M$  and subjecting it to a maximum entropy criterion, restricted by the constraints imposed by Hermiticity and the symmetries. It turns out that in this maximally random distribution, the matrix elements are uncorrelated, i.e.  $P(\{M_{ij}\}) = \prod_{ij} P(M_{ij})$ , and have a Gaussian-like distribution.

It is often quite useful to deal directly with the evolution operator generated by the Hamiltonian instead of the Hamiltonian itself. This is specially true when the Hamiltonian of the system is time dependent and periodic. Then, one defines the Floquet operator which maps the state of the system from one oscillation period to the next. Unitary operators too can be constrained by symmetries. Hence, in a very similar fashion as for Hermitian matrices, one can derive a maximally random probability distribution over the set of unitary operators restricted by some symmetries.

With these tools in hand, we can compute the average value of some quantities over the set of Hamiltonians<sup>1</sup> satisfying some symmetries. Thus, we seek for statistical signatures distinguishing the Hamiltonians with different symmetries; these differences may be with respect to the nature of the symmetries or their numbers. In particular, we are interested in determining whether the system possesses many symmetries — roughly as many as its number of degrees of freedom — and hence is integrable or if it possesses just a few symmetries, which we expect to yield chaotic dynamics in the classical limit.

As mentioned in the first section, these techniques can also be used to study complex systems with just a few degrees of freedom: quantized deterministic chaos. The random matrix conjecture asserts that some statistical properties of quantized chaotic systems are typically well modeled by those of random matrices with appropriate symmetries. While many important mathematical results underpin the

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<sup>1</sup>Here and henceforth, we will refer to Hamiltonians to address issues which apply equally well to constant Hamiltonians and time dependent periodic Hamiltonians, hence to Floquet operators.

conjecture, a rigorous proof is lacking and support rests on a very large accumulation of numerical results.

### Energy distribution

Perhaps the most famous of all “quantum signatures of chaos” is the level spacing distribution. Before discussing it in detail, let us digress to generalities about energy statistics. The eigenvalues of a Hamiltonian represent the energies of the system, each energy is associated with an eigenstate or eigensubspace in the case where it is degenerated<sup>2</sup>. It follows from Hermiticity that these eigenstates/subspaces are perfectly distinguishable. Thus, a Hermitian matrix can be parametrized in terms of its “energies” and their associated states. This allows to go from the probability distribution over matrix elements  $P(\{M_{ij}\})dM$  to a distribution over energies and “angles” in Hilbert space  $P(\{E_i, \{\vec{\theta}_i\})J(\{E_i, \{\vec{\theta}_i\})dEd\vec{\theta}$  where  $J$  is the appropriate Jacobian. It turns out that this Jacobian is always separable in energies and angles and so is the probability distribution in the case of maximal randomness. This indicates that for those random matrices, the eigenvectors and eigenvalues are uncorrelated. Thus, integrating over the angles only yields a multiplicative constant and one is left with the marginal distribution over energies  $P(\{E_i\})$ .

### Level spacing distribution

The probability distribution over energies will allow us to compute the mean level spacing distribution over a set of random matrices. The level spacing distribution of a given Hamiltonian is a profile of the nearest neighbor distance of energy levels. Assuming that the eigenvalues  $E_i$  are arranged in increasing order, the level spacing distribution is the distribution of the variable  $S = E_{i+1} - E_i$ . It is thus a simple mathematical exercise (!) to compute the mean distribution of this variable over a set of random Hermitian matrices constrained by some symmetries:  $P(S) = \int (E_{i+1} - E_i)\delta(s - [E_{i+1} - E_i])P(\{E_i\})dE$ . The outcome of this integration is astonishing: when the Hamiltonian possesses many symmetries, the resulting distribution is Poissonian; for Hamiltonians with but a few symmetries, it follows some low power law before decreasing exponentially for large values of  $S$ .<sup>3</sup> The exact details of the power law depends in the nature and number of symmetries imposed on the set, but these details are not important. What is crucial is that

<sup>2</sup>For Floquet operators, the following applies to eigenphases or quasi-energies: the phases of the complex eigenvalues of a unitary operator

<sup>3</sup>Actually, this result is obtained when the average is carried over an invariant subspace of the symmetries, not the entire energy spectrum; this technical detail, although important, is not relevant to the current discussion.

for systems with many symmetries, the distribution reaches a maximum when  $S$  approaches zero while it reaches a minimum for systems with no or just a few symmetries. In short, quantized chaotic systems tend to exhibit *level repulsion* while quantized regular systems typically show *level clustering*.

Level repulsion can be understood from a simple fact: no level crossing without symmetries. Assume that the Hamiltonian depends on some parameter  $\lambda$ , for example,  $\lambda$  could be the strength of a perturbation. Of course, the eigenvalues of  $H(\lambda)$  also depend on this parameter. As the value of  $\lambda$  is varied, the energies levels will change but cannot cross each other unless there is a symmetry in the Hamiltonian. Thus, the tendency of energy levels to avoid approaching at a short distance from one another is a consequence of the lack of symmetries of chaotic systems: the absence of symmetries forces the energy levels to be correlated in order to avoid each other.

For integrable systems on the other hand, the presence of many symmetries allows one to write the Hamiltonian in a block diagonal way; one block per invariant subspace. This fact can be seen as the direct and only consequence of the imposed symmetries: two states  $|i\rangle$  and  $|j\rangle$  associated to different eigenvalues of the symmetries cannot have a finite matrix element  $\langle i|H|j\rangle$  if  $H$  satisfies these symmetries. Hence, we know that  $H$  must be block diagonal. As a consequence of the maximum entropy requirement, these blocks are statistically independent. Within a block, the energy distribution has a certain structure, energy levels are somehow correlated. Nevertheless, when looking at the spectrum of the entire Hamiltonian, one is not distinguishing the levels belonging to different blocks: a large number of levels associated to statistically independent blocks pile up, washing out the existing inner-block correlation. It is thus not surprising that the emerging spacing distribution is Poissonian, a characteristic of *uncorrelated* random variables.

While all these statistical predictions rely on the random matrix conjecture, they are in overwhelming agreement with experiments, either numerical or “physical”. Experimental results however, cannot be *directly* compared to those predictions. Indeed, these predictions are established for an ensemble average, there is no reason to expect that a single instance of an Hamiltonian should reproduce this statistical mean. Thus, we need to simulate the effect of an ensemble average. For physical experiments, the most convenient way of simulating this average is to convolute the energy spectrum with some normalized tunable-width function, e.g. a Gaussian. The width of this function is fixed by a self consistency test: the smoothed density should not vary considerably over an interval of the order of the mean level spacing, furthermore it should be quite larger than the mean level spacing so the average is taken over many levels. For numerical experiments, the ensemble average can be

reproduced by averaging over some parameter of the Hamiltonian on a conveniently chosen range. In our previous example, one could average the energy spectrum over increasing ranges of  $\lambda$  until the fluctuations on a scale of the mean level spacing vanish, i.e. till the effect of the discreteness of the energy levels disappear.

### Form factors and secular coefficients

There is a huge amount of literature devoted to the connection between random matrix theory and quantum chaos, our goal here is obviously not to give a fair account of this field but rather to relate its general framework. Thus, we shall introduce two last spectral properties of Floquet operators. One could also study these properties for a time independent Hamiltonian by suitably choosing a time  $T$  and computing the associated unitary time evolution operator for this period. This time should be chosen so the maximal action is roughly equal to  $2\pi$ . This is to make sure that the eigenphases are nicely distributed over the entire complex circle while at the same time, are not “wrapped around it” too many times; this would fade out the existing correlations.

The form factor  $T_n$  of a Floquet operator  $F$  is simply the modulus square of the trace of  $F$  to the power  $n$ :  $T_n = |\text{Tr}\{F^n\}|^2$ . In the jargon of random matrix theory, the secular coefficients  $a_n$  are the coefficients of the characteristic polynomial of a Floquet operator:  $\text{Det}\{F - x\mathbb{1}\} = \sum_n^N a_n x^n$  where  $N$  is the dimension of  $F$ . It may seem curious that form factors and secular coefficients are intimately related, the knowledge of the form factors allows one to compute the secular coefficients and vice versa. In fact, this is no mystery: the first  $N/2$  values of either the form factor or the secular coefficients uniquely determines the quasi-energy spectrum. On the other hand, it is obvious how to compute any  $T_n$  or  $a_n$  given that spectrum.

Thus, one might legitimately doubt that these functions, so intimately related to the details of the eigenphases structure, can serve as indicators of universal features of symmetries classes; nevertheless they do! First, let’s guess what the ensemble average of these functions should yield for regular and chaotic systems, we will focus on the form factors.  $T_1$  is simply the modulus square of the trace of the Floquet operator, i.e. of the sum of its eigenvalues. As argued above, the large number of conserved quantities of regular systems washes out all correlations between quasi-energies — the quasi-energies of an integrable system are random complex numbers of unit length. The sum of  $N$  of these numbers is thus analog to a random walk in the plane, each step having unit length but random orientation. We can thus use standard results of random walk to conclude that for regular systems, the ensemble average of  $T_1$  is  $N$ . For larger values of  $n$ , the same argument holds. If the angles  $\phi_j$  — eigenphases of  $F$  — are statistically independent, so are the angles

$n\phi_j \bmod 2\pi$  — eigenphases of  $F^n$ . We conclude that the ensemble average of the form factor  $T_n$  of regular systems are equal to  $N$ , independently of  $n$ .

For chaotic systems, there is no such simple argument. Nevertheless, using the random walk analogy, it becomes clear that the average value of  $T_n$  for small  $n$  should be less than what it was for regular systems. Indeed, because of level repulsion, the random walk is biased; each step tend to be taken in a direction anti-correlated with the other steps. As a result, the distance from the origin after  $N$  steps should be less than  $\sqrt{N}$ , the result obtained for statistically independent steps. As  $n$  increases, the angles  $n\phi_j \bmod 2\pi$  show less and less correlations until they become completely independent,  $T_n$  should thus increase with  $n$  before saturating at the value of unbiased walk. A more sophisticated calculation shows that  $T_n$  should be roughly proportional to  $n$  before saturating at  $N$ , the value obtained for the uncorrelated eigenphases of regular systems.

As argued above, one should not expect a fixed Floquet operator to reproduce these ensemble averages. Nevertheless, there are quite elegant techniques to extract these universal features from a given system. If the system possesses a free parameter  $\lambda$ , one can, once again, average the spectral function over some range of  $\lambda$  to simulate an ensemble average. Needless to say, this technique is mostly restricted to numerical studies. In the laboratory, the coupling strength and other parameters are usually fixed. A clever trick to solve this problem is to use a sort of ergodic theorem, quite similar to the one used in thermodynamics. In short, this theorem states that the time average of some quantities can reproduce, under some conditions, an ensemble average. Moreover, the fluctuations from the ensemble mean drop down as the inverse of the time averaging range. Here,  $n$  plays the role of time, so the theorem tell us that a suitable average of  $T_n$  over a range of  $n$  should reproduce quite accurately the universal features of its symmetry class. These are examples illustrating how random matrix theory can be a useful tool to study unknown symmetries and hence the emergence of chaos.

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## Reading guide

There is a large amount of literature devoted to quantum chaos, much too large to give it a fair account here. We will simply give references to what was used by the author as “starting points” from which the rest of the literature can be explored.

- M.V. Berry and M. Tabor, *Level clustering in the regular spectrum*, *Proc. R. Soc. London A* **356**, 375 (1977).  
Contains a proof that the level spacing distribution of regular systems should follow a Poisson law.
- J.V. Emerson, *Chaos and Quantum-Classical Correspondence for two Coupled Spins*, Ph.D. thesis, Simon Fraser University, arXiv: quant-ph/0211035 (2001).  
A good exposition of the quantum-classical correspondence problem which emerges from chaotic systems. Presents evidences which indicate that Liouville’s mechanics is in better agreement with the predictions of quantum mechanics and makes it an argument for the interpretation of the wave function.
- F. Haake, *Quantum Signatures of Chaos*, Springer-Verlag, New York, 2nd Ed. (2000).  
A technical book on quantum chaos which addresses most topics covered in the present tutorial. Emphasizes on the relation to random matrix theory. It also contains Furtenberg’s theorem used in this manuscript.
- A. Peres, *Quantum Theory: Concepts and Methods*, Kluwer academic publisher, Boston (1993).  
Presents quantum chaos in terms of extreme sensitivity to perturbation. Contains many numerical examples which are explained with a strong intuition.
- R. Schack and C.M. Caves, *Information-theoretic characterization of quantum chaos*, *Phys. Rev. A* **53**, 3257 (1996).  
Introduces the tradeoff between Landauer’s erasure and work extraction as a characterization of quantum chaos.
- H.J. Stöckmann, *Quantum Chaos: an Introduction*, Cambridge University Press, Cambridge, (1999).  
Also a very general book on quantum chaos covering many topics. Differs from Haake’s book with an intuitive physicist approach.

- W.H. Zurek and J.P. Paz, *Quantum chaos: a decoherent definition*, *Physica D* **83**, 300 (1995).

It is shown how decoherence can reestablish the quantum-classical correspondence. It is also postulated that the increase of a system's entropy under interaction with an environment can serve as a signature of chaos.