

Toy Model for a Relational Formulation of Quantum Theory

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In the absence of an external frame of reference—i.e., in background independent theories such as general relativity—physical degrees of freedom must describe *relations* between systems. Using a simple model, we investigate how such a relational quantum theory naturally arises by promoting reference systems to the status of dynamical entities. Our goal is twofold. First, we demonstrate using elementary quantum theory how any quantum mechanical experiment admits a purely *relational* description at a fundamental. Second, we describe how the original “non-relational” theory approximately emerges from the fully relational theory when reference systems become semi-classical. Our technique is motivated by a Bayesian approach to quantum mechanics, and relies on the noiseless subsystem method of quantum information science used to protect quantum states against undesired noise. The relational theory naturally predicts a fundamental decoherence mechanism, so an arrow of time emerges from a time-symmetric theory. Moreover, our model circumvents the problem of the “collapse of the wave packet” as the probability interpretation is only ever applied to diagonal density operators. Finally, the physical states of the relational theory can be described in terms of “spin networks” introduced by Penrose as a combinatorial description of geometry, and widely studied in the loop formulation of quantum gravity. Thus, our simple bottom-up approach (starting from the semiclassical limit to derive the fully relational quantum theory) may offer interesting insights on the low energy limit of quantum gravity.

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1. INTRODUCTION

To combine the two main physical theories of the twentieth century—quantum mechanics and general relativity—it is important to clearly identify the chief insights they offer on the physical world. Quantum mechanics establishes a mathematical apparatus—Hilbert space, canonical quantization, etc.,—that sets a general framework to describe physical systems. Here, we will assume that this general framework is essentially correct. The main lesson retained from

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general relativity is that physical theories should *not* be formulated in terms of a background reference frame, but rather should be *relational*; a point of view emphasized by Rovelli (1996, 2004) among other. Starting from plain elementary quantum mechanics, we investigate consequences of background independence.

More precisely, we will argue that an orthodox (non-relational) physical description can be made purely relational by following four simple guidelines. (Here and throughout this manuscript, we call “orthodox” the description of a system found in any elementary textbook on quantum mechanics, where time and space are treated as absolute external parameters.) These guidelines are:

- (1) Treat everything quantum mechanically.
- (2) Use Hamiltonians with appropriate symmetries.
- (3) Introduce equivalence classes between quantum states related by an element of the symmetry group.
- (4) Interpret diagonal density operators as probability distributions.

In the appropriate semiclassical limits, the relational theory will be equivalent to the orthodox quantum description, in the sense that it leads to the same physical predictions. However, away from these limiting regimes, the relational theory also predicts new phenomenon, such as a fundamental decoherence process (Gambini *et al.*, 2004).

Our first guideline suggests that we should not resort to semiclassical approximations in the description of a physical system. Such approximations are often responsible for breaking the symmetries of the system, and in effect, introduce a background reference frame. For example, treating an external magnetic field as classical provides a natural axis to quantize angular momentum. Note that quantum theory is not required to arrive at a relational description: there are perfectly valid relational classical theories. In a classical relational theory, it is often possible to choose an arbitrarily system as a reference frame and recover a nonrelational theory, e.g., by working in the rest frame of a specific particle. In a quantum settings however, switching from the relational to the nonrelational description will always require some level of *approximation* since reference frames defined with respect to quantum systems are subject to quantum fluctuations (Aharonov and Kaufherr, 1984; Rovelli, 1991; Toller, 1997; Mazzucchi, 2000). These quantum effects can be made arbitrarily small by increasing the mass of the reference system. However, in practical situations, reference systems have finite masses. Moreover, increasing their masses will induce gravitational distortions to the measured quantities (Wigner, 1957; Giovannetti *et al.*, 2004), so it is essential to keep the masses finite. Here, we will neglect gravitational corrections and focus on quantum mechanical effects.

The second guideline follows as a corollary of the first. Once every physical system—including reference systems—are treated quantum mechanical, the fundamental Hamiltonians will have the appropriate symmetries for the obvious

reason that it is not possible to write down a symmetry breaking term without use of external coordinates.

The third guideline is used to get rid of unphysical information in the description of the system. States that are related by a transformation that belongs to the symmetry group of the system should be regarded as physically equivalent. This point of view is quite natural if one adopts a Bayesian interpretation of quantum states (Caves *et al.*, 2002; Fuchs, 2002). Following a Bayesian prescription, the lack of an external reference frame leads to group averaging of the quantum states. The effect of this group average will be to randomize the unphysical degrees of freedoms—those defined with respect to an external reference frame—while leaving the relational (hence physical) ones unchanged. At this stage, the unphysical degrees of freedom can be removed from the description as they carry no information. This procedure is inspired by the noiseless subsystems method (Knill *et al.*, 2000; Zanardi, 2001; Kempe *et al.*, 2001) used to protect the state of “virtual” quantum systems in quantum information science. In this language, the physical degrees of freedom form noiseless subsystems of a noise algebra, where the noise operators are elements of the symmetry group of the system. At a formal level, the third guideline says that we must quotient the state space of physical systems by their symmetry group. We note that the Bayesian prescription differs from the “coherent” group average commonly encountered in quantum gravity (see Marolf, 2000), leading to distinct physical descriptions.

Finally, guideline 4 gives the standard interpretation to quantum states, but circumvents the problematic collapse of the wave packet. All four guidelines will get clarified as we apply them to a simple example in the next section.

The outline of the paper is as follows. In the next section, we will present the simplest textbook quantum mechanical system: a spin- $\frac{1}{2}$ particle immersed in a uniform constant magnetic field. Despite its simplicity, the orthodox description of this system violates all four guidelines, so it provides a good starting point to illustrate our procedure. In the following Sections (2.1, 2.2, and 2.3), we gradually apply our procedure to this system and arrive at a relational theory that is equivalent to the original theory in the appropriate “macroscopic” limit. Away from these limits however the relational theory predicts new phenomenon such as a fundamental decoherence mechanism, that can clearly be seen in our numerical analysis. In Section 3, we describe the general picture that derives from our four guideline. The main ingredients of the construction are the noiseless subsystems of quantum information science that will be briefly described. Section 4.1 discusses the fundamental decoherence mechanism that arises from the relational theory. In short, decoherence will occur whenever clocks are of finite size and therefore subject to quantum fluctuations (Gambini *et al.*, 2004), leading to an arrow of time in a time-independent theory. Section 4.2 establishes a connection between the basis states of the relational theory and “spin networks” introduced by Penrose (1971) as a combinatorial description of geometry and widely studied in the loop

formulation of quantum gravity (Rovelli and Smolin, 1995; Baez, 1995) (see also Major, 1999). It is our hope that the slight distinction between how spin networks arise in our “semiclassically inspired” model and how they are used in loop quantum gravity will yield some new insights on the low energy regime of quantum gravity. In Section 4.3, we speculate about possible extensions of the program and relate it to the concept of superselection rule and spontaneous symmetry breaking. Finally, we conclude with a summary in Section 5.

2. AN EXAMPLE

We begin by illustrating our program with the simplest quantum mechanical system: the system (\mathcal{S}), a spin- $\frac{1}{2}$ particle, is interacting with a uniform magnetic field \vec{B} . In this toy Universe, the spacetime manifold has the topology $S_2 \times S_1$; there are only orientations in 3-space (hence the 2-sphere) and time (which we assume takes a finite range with periodic boundary conditions, and hence has the topology of a circle). Accordingly, the fundamental symmetries are $SO(3)$ and $U(1)$. The orthodox description of this system goes as follows. Without loss of generality, we assume that \vec{B} is along the x axis, so the system’s Hamiltonian is $H^{\mathcal{S}} = -B\sigma_x^{\mathcal{S}}$ where all physical constants are absorbed in B . The system’s initial conditions are specified by the state $|\psi(0)\rangle^{\mathcal{S}} = \alpha|\uparrow\rangle^{\mathcal{S}} + \beta|\downarrow\rangle^{\mathcal{S}}$, where the quantization axis is arbitrarily chosen to be along the z direction (this will be the case throughout this manuscript, unless specified otherwise). At time t , the state of the system is

$$|\psi(t)\rangle^{\mathcal{S}} = \alpha(t)|\uparrow\rangle^{\mathcal{S}} + \beta(t)|\downarrow\rangle^{\mathcal{S}}, \tag{1a}$$

$$\alpha(t) = \alpha \cos(Bt/2) + i\beta \sin(Bt/2) \tag{1b}$$

$$\beta(t) = i\alpha \sin(Bt/2) + \beta \cos(Bt/2). \tag{1c}$$

If we wish to measure the value of the spin of the system at time τ , say along the z -axis, we must introduce a measurement apparatus \mathcal{A} that couples to \mathcal{S} . The time-dependent interaction Hamiltonian

$$H^{\mathcal{S}\mathcal{A}}(t) = -g\delta(t - \tau)\sigma_z^{\mathcal{S}} \otimes \sigma_y^{\mathcal{A}} \tag{2}$$

is a good choice of “measurement” coupling. The coupling constant is set to $g = 2\pi$. Given the initial state of the apparatus $(|\uparrow\rangle^{\mathcal{A}} + |\downarrow\rangle^{\mathcal{A}})/\sqrt{2}$, the joint state of $\mathcal{S} - \mathcal{A}$ at time $t = \tau_+$ immediately after the interaction is

$$|\Psi(\tau_+)\rangle^{\mathcal{S}\mathcal{A}} = \alpha(\tau)|\uparrow\rangle^{\mathcal{S}} \otimes |\uparrow\rangle^{\mathcal{A}} + \beta(\tau)|\downarrow\rangle^{\mathcal{S}} \otimes |\downarrow\rangle^{\mathcal{A}}.$$

This pre-measurement phase establishes correlations between \mathcal{S} and \mathcal{A} . The next step of the measurement process is the collapse of the wave function, which asserts that the measurement apparatus, being a “classical” object, cannot be in a quantum

superposition, so it rapidly collapses into either the up or down state, each with probabilities given by amplitude squared. In certain circumstances, this step can be given an operational justification (Zurek, 2003; Ollivier *et al.*, 2004; Poulin, 2004). At this stage, the pair $\mathcal{S} - \mathcal{A}$ is described by the mixed state

$$\rho^{\mathcal{S}\mathcal{A}} = |\alpha(\tau)|^2 |\uparrow\rangle\langle\uparrow|^{\mathcal{S}} \otimes |\uparrow\rangle\langle\uparrow|^{\mathcal{A}} + |\beta(\tau)|^2 |\downarrow\rangle\langle\downarrow|^{\mathcal{S}} \otimes |\downarrow\rangle\langle\downarrow|^{\mathcal{A}}.$$

The interpretation of this state is that both \mathcal{S} and \mathcal{A} are either in the up state with probability $|\alpha(\tau)|^2$, or both in a down state with probability $|\beta(\tau)|^2$, given by (1), and this completes the measurement process.

This description is obviously not background independent as it makes explicit use of an external coordinate system. In the case of the external field, this dependence is explicit: $\vec{B} \propto \hat{x}$, where \hat{x} makes reference to a coordinate system. The dependence of the measurement apparatus on an external reference frame is twofold. First, the coupling Hamiltonian (2) used to establish correlation between system and apparatus is neither rotationally nor time-translationally invariant, thus conflicting with guideline 2. Moreover, the collapse phase requires the specification of a preferred observable: classical objects cannot be in superposition involving different values of this preferred observable. We say that the preferred observable is *superselected*. In the above example, the preferred observable was the angular momentum along the z -axis, once again making reference to an external coordinate system.

We will refer to the example presented in this section as the “toy model.” In the next subsections, we will apply our guidelines to the toy model and eliminate the need for an external reference frame, demonstrating how one naturally arrives at a purely relational theory. Before doing so, we must pause to establish some notation. In what follows, we use several particle with different angular momentum (or spin) to describe the toy model. To avoid confusion, we adopt the following notation. Each particle is given a name that is represented by a calligraphic capital letter, e.g., \mathcal{A} . Operators, states, and Hilbert space referring to this particle will have the associated letter as a superscript. The quantum number associated to the total angular momentum of the particle is represented by the same capital letter in roman fonts, while the quantum number for the z component will be labeled by the lower case letter. For particle \mathcal{A} , this gives $(J^{\mathcal{A}})^2|A, a\rangle^{\mathcal{A}} = A(A + 1)|A, a\rangle^{\mathcal{A}}$ and $J_z^{\mathcal{A}}|A, a\rangle^{\mathcal{A}} = a|A, a\rangle^{\mathcal{A}}$, with $|A, a\rangle^{\mathcal{A}} \in \mathcal{H}^{\mathcal{A}} = \mathbb{C}^{2A+1}$. Exception will be made for spin- $\frac{1}{2}$ particles where the up and down arrows are used. Finally, we denote $\mathcal{B}(\mathcal{H})$ the set of bounded linear operators acting on \mathcal{H} .

2.1. Measurement

Our goal is now to demonstrate how our guidelines naturally lead to a background independent relational theory that is equivalent to the orthodox description *in the appropriate limits*. Hence, in this section and the following, we will often

be interested in various limiting regimes of the relational theory. These limits are not constitutive to the theory: their sole purpose is to demonstrate compatibility with known regimes. Of course, the new and interesting physics will arise when the relational theory is analyzed away from these limits.

Let us first assume that there is no system Hamiltonian, so the system's state is $\alpha|\uparrow\rangle^S + \beta|\downarrow\rangle^S$ at all times. To perform a spin measurement in the absence of an external reference frame, we need a *gyroscope* \mathcal{G} . Following guideline 1, we should treat this gyroscope quantum mechanically. A good choice consists of a spin- G particle² with large value of G , prepared in a state of maximal angular momentum along the z direction, i.e., in the quantum state $|G, g = G\rangle^{\mathcal{G}}$, which we abbreviate $|G, G\rangle^{\mathcal{G}}$. States of maximum angular momentum along a certain axis, also called $SU(2)$ coherent states, are appreciated for their semiclassical properties, and as such, they will be used extensively here to recover the nonrelational limit. Note that the description of a gyroscope relies on an external coordinate system, but we will soon get rid of it.

At any given time, the joint state of the system and the gyroscope is thus

$$|\Psi\rangle^{S\mathcal{G}} = (\alpha|\uparrow\rangle^S + \beta|\downarrow\rangle^S) \otimes |G, G\rangle^{\mathcal{G}}. \tag{3}$$

As noted above, this state describes unphysical degrees of freedom as it is defined relative to a nonexistent coordinate system. To eliminate this pathology, we follow guideline 3 and introduce equivalence classes between states in the composite Hilbert space $\mathcal{H}^{S\mathcal{G}} = \mathbb{C}^2 \otimes \mathbb{C}^{2G+1}$. For this, we represent the quantum state of (3) by the density operator $\rho^{S\mathcal{G}} = |\Psi\rangle\langle\Psi|^{S\mathcal{G}} \in \mathcal{B}(\mathcal{H}^{S\mathcal{G}})$. The equivalence classes are obtained by applying the trace preserving completely positive (TPCP) map $\mathcal{E}^{S\mathcal{G}} : \mathcal{B}(\mathcal{H}^S \otimes \mathcal{H}^{\mathcal{G}}) \rightarrow \mathcal{B}(\mathcal{H}^S \otimes \mathcal{H}^{\mathcal{G}})$ defined by the action

$$\mathcal{E}^{S\mathcal{G}}(\rho) = \int_{SO(3)} R^{S\mathcal{G}}(\Omega)\rho R^{S\mathcal{G}}(\Omega)^\dagger d\Omega, \tag{4}$$

where $R^{S\mathcal{G}} = R^S \otimes R^{\mathcal{G}}$ is the unitary representation of the rotation group on the pair $\mathcal{S} - \mathcal{G}$, and $d\Omega$ is the invariant Haar measure on $SO(3)$. The resulting state $\rho^{S\mathcal{G}} = \mathcal{E}^{S\mathcal{G}}(\rho^{S\mathcal{G}})$ is thus rotationally invariant. The map \mathcal{E} generalizes to any number of particles in an obvious manner.

The representation $R^{S\mathcal{G}}$ is generated by the *total* angular momentum operator $\vec{J}^{S\mathcal{G}} = \vec{\sigma}^S + \vec{J}^{\mathcal{G}}$, where $\vec{\sigma}^S = (\sigma_x^S, \sigma_y^S, \sigma_z^S)$ and $\vec{J}^{\mathcal{G}} = (J_x^{\mathcal{G}}, J_y^{\mathcal{G}}, J_z^{\mathcal{G}})$ are the system's and gyroscope's angular momentum operators respectively. The representation therefore commutes with the operators $(J^{S\mathcal{G}})^2 = \vec{J}^{S\mathcal{G}} \cdot \vec{J}^{S\mathcal{G}}$, $(\sigma^S)^2$ and $(J^{\mathcal{G}})^2$. Hence, to study the effect of $\mathcal{E}^{S\mathcal{G}}$, it is useful to express $|\Psi\rangle^{S\mathcal{G}}$ in terms of

²Note that the gyroscope can be a composite particle. For example, the gyroscope could be a ferromagnet composed of 10^{23} spin- $\frac{1}{2}$ particles all roughly aligned in the same direction, which for our purposes behaves as a single particle with large spin.

the total angular momentum:

$$\begin{aligned}
 |\Psi\rangle^{SG} &= \alpha|G + \frac{1}{2}, G + \frac{1}{2}; \frac{1}{2}; G\rangle + \frac{\beta}{\sqrt{2G+1}}|G + \frac{1}{2}, G - \frac{1}{2}; \frac{1}{2}; G\rangle \\
 &\quad + \frac{\beta\sqrt{2G}}{\sqrt{2G+1}}|G - \frac{1}{2}, G - \frac{1}{2}; \frac{1}{2}; G\rangle.
 \end{aligned}$$

Above, we use standard angular momentum notation: $|j, m; j_1; j_2\rangle$ is a simultaneous eigenstate of $(J^{SG})^2$, J_z^{SG} , $(\sigma^S)^2$, and $(J^G)^2$ with eigenvalues $j(j+1)$, m , $j_1(j_1+1)$, and $j_2(j_2+1)$ respectively (see e.g., Sakurai, 1994). As J_z^{SG} is the only operator defined with respect to the external reference frame, the effect of \mathcal{E} on this state can be readily anticipated: it randomizes the value of m while leaving the other quantum numbers unaffected (Bartlett *et al.*, 2004). Indeed, the expression we get is

$$\begin{aligned}
 \mathcal{E}^{SG}(\rho^{SG}) &= \left[|\alpha|^2 + \frac{|\beta|^2}{2G+1} \right] |G + \frac{1}{2}; \frac{1}{2}; G\rangle\langle G + \frac{1}{2}; \frac{1}{2}; G| \otimes \\
 &\quad \frac{\mathbb{1}_{2G+2}}{2G+2} + \frac{2G|\beta|^2}{2G+1} |G - \frac{1}{2}; \frac{1}{2}; G\rangle\langle G - \frac{1}{2}; \frac{1}{2}; G| \otimes \frac{\mathbb{1}_{2G}}{2G},
 \end{aligned}$$

where the identity operators $\mathbb{1}$ act on the J_z^{SG} sectors. Thus, the unphysical degree of freedom associated with J_z^{tot} is now in a maximally mixed state and can be removed from the physical description to arrive at

$$\begin{aligned}
 \rho_{\text{physical}}^{SG} &= \left[|\alpha|^2 + \frac{|\beta|^2}{2G+1} \right] |G + \frac{1}{2}; \frac{1}{2}; G\rangle\langle G + \frac{1}{2}; \frac{1}{2}; G| \\
 &\quad + \frac{2G|\beta|^2}{2G+1} |G - \frac{1}{2}; \frac{1}{2}; G\rangle\langle G - \frac{1}{2}; \frac{1}{2}; G|. \tag{5}
 \end{aligned}$$

Guideline 4 gives the desired interpretation to this state: a spin- G and a spin- $\frac{1}{2}$ particle (here the gyroscope and the system respectively) are either parallel, yielding a total angular momentum $G + \frac{1}{2}$, or antiparallel, yielding $G - \frac{1}{2}$. The probabilities associated to these two alternatives are $P(\text{parallel}) = |\alpha|^2 + |\beta|^2/(2G+1)$ and $P(\text{antiparallel}) = 2G|\beta|^2/(2G+1)$. When the gyroscope is of a macroscopic size, we recover the familiar probabilities $|\alpha|^2$ and $|\beta|^2$. Note that these probabilities are the diagonal entries of a diagonal density matrix: no collapse was required to recover the probability rule.

The general picture illustrated here for a spin- $\frac{1}{2}$ particle holds for an arbitrary spin. This follows from the fact that the Clebsch–Gordan coefficients

$C(S, s; G, g; J, j) = (\langle S, s | \otimes \langle G, g | | J, m; S, G \rangle$ satisfy³

$$\lim_{G \rightarrow \infty} |C(S, s; G, G; G + s + \Delta, G + s)|^2 = \begin{cases} 1 & \text{if } \Delta = 0 \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

Thus, given a spin- S system in state $|\psi\rangle^S = \sum_s \alpha_s |S, s\rangle$ and a gyroscope in state $|G, G\rangle^G$, we can write the combined state in terms of the total angular momentum as

$$|\Psi\rangle^{SG} = \sum_{s, J} \alpha_s C(S, s; G, G; J, G + s) |J, G + s; S; G\rangle.$$

When the gyroscope reaches macroscopic scales, this state approaches $\sum_s \alpha_s |G + s, G + s; S; G\rangle$ by virtue of (6). Guideline 3 thus implies

$$\rho_{\text{physical}}^{SG} \approx \sum_s |\alpha_s|^2 |G + s; S; G\rangle \langle G + s; S; G|. \quad (7)$$

This state has the desired interpretation: the angle θ between the gyroscope and the system satisfies $\cos \theta = s/S$ with probability $|\alpha_s|^2$.

2.2. Dynamics

By promoting the reference frame to the status of a dynamical entity, a gyroscope, we have demonstrated how the quantum probability rule is recovered in a macroscopic limit of a background independent theory (see also e.g., Aharonov and Kaufherr, 1984; Rovelli, 1991; Toller, 1997; Mazzucchi, 2000). The next step is to introduce nontrivial dynamics. The only symmetric single-particle Hamiltonian is trivial, so following guideline 2, dynamics must be caused by interaction. In the toy model, this was achieved by applying an external magnetic field \vec{B} that made the spin precess. To model this field, all we need is a big magnet \mathcal{M} . Following guideline 1, this magnet must be quantum mechanical, so we represent it with a particle of maximal spin along the x axis:

$$|M, M\rangle_x^{\mathcal{M}} = \frac{1}{2^M} \sum_{m=-M}^M \binom{2M}{M+m}^{1/2} |M, m\rangle^{\mathcal{M}}.$$

We must couple this magnet to the system with a symmetric Hamiltonian, so it has to be a scalar function of $\vec{J}^{\mathcal{M}} \cdot \vec{\sigma}^S$. We choose the Heisenberg coupling $H^{S\mathcal{M}} = -2\lambda \vec{J}^{\mathcal{M}} \cdot \vec{\sigma}^S$. Given the system's initial state $\alpha|\uparrow\rangle + \beta|\downarrow\rangle$, we can easily

³ We have verified this limit numerically up to accuracy roughly 1% for G of a few hundreds, but have not been able to derive it analytically.

solve the equation of motion and get

$$\begin{aligned}
 |\Psi(t)\rangle^{\mathcal{SM}} = & |M, M\rangle_x^{\mathcal{M}} \otimes |\psi(t)\rangle^{\mathcal{S}} + C(t) \left[\frac{1}{\sqrt{2M}} |M, M\rangle_x^{\mathcal{M}} \otimes |\downarrow\rangle^{\mathcal{S}} \right. \\
 & \left. + |M, M-1\rangle_x^{\mathcal{M}} \otimes |\uparrow\rangle^{\mathcal{S}} \right] \tag{8}
 \end{aligned}$$

where $|\psi(t)\rangle^{\mathcal{S}}$ is the solution to the toy model given by (1) with $B = \lambda(2M + 1)$, and the function $C(t)$ is equal to $i\sqrt{M}2(\alpha - \beta)\sin(Bt/2)/(2M + 1)$. We see that $C(t) \sim 1/\sqrt{M}$, so when the magnet reaches macroscopic sizes, we obtain the same formal solution as we did with the toy model. Note however that the physical description is not yet rotationally invariant as states are expressed with respect to an external z quantization axis.

Once again, what we have illustrated here with a spin- $\frac{1}{2}$ system is true in general and follows from (6) and another similar identity

$$\lim_{M \rightarrow \infty} |C(S, s + \Delta; M, M - \Delta; M + s, M + s)|^2 = \begin{cases} 1 & \text{if } \Delta = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Taking the limit $M \rightarrow \infty$ while keeping $2\lambda M = B$ will result in the state

$$|\Psi(t)\rangle^{\mathcal{SM}} \approx \sum_s \alpha_s e^{iBst} |S, s\rangle_x^{\mathcal{S}} |M, M\rangle_x^{\mathcal{M}} \tag{10}$$

as expected for a spin- S particle immersed in a magnetic field along the x -axis.

To eliminate the unphysical reference frame from the above discussion, we need to reintroduce the gyroscope and apply guideline 3. At this stage, only relational degrees of freedom between \mathcal{S} , \mathcal{M} , and \mathcal{G} will remain. One can easily recover the nonrelational result by letting both the gyroscope and the magnet reach macroscopic sizes. After a few algebraic manipulations and keeping only the nonvanishing terms, we arrive at

$$\rho_{\text{physical}}^{\mathcal{SMG}} \approx \frac{1}{2^{2M}} \sum_{n=-M}^{M-1} \binom{2M}{M+n} |\Psi_n(t)\rangle \langle \Psi_n(t)|^{\mathcal{SMG}} \tag{11}$$

where we have defined

$$\begin{aligned}
 |\Psi_n(t)\rangle^{\mathcal{SMG}} = & \alpha(t) |G + \frac{1}{2} + n; G + \frac{1}{2}\rangle^{\mathcal{SMG}} \\
 & + \beta(t) \sqrt{\frac{M-n}{M+n+1}} |G + \frac{1}{2} + n; G - \frac{1}{2}\rangle^{\mathcal{SMG}} \tag{12}
 \end{aligned}$$

where α and β are defined in the toy model, cf. (1). In the above equation, the quantum numbers refer to the eigenvalues of the total angular momentum, and the joint

angular momentum of the system and the gyroscope, i.e., $(J^{SMG})^2|a; b\rangle^{SMG} = a(a + 1)|a; b\rangle^{SMG}$ and $(J^{SG})^2|a; b\rangle^{SMG} = b(b + 1)|a; b\rangle^{SMG}$. These quantum numbers are purely relational. The quantum numbers associated to $(\sigma^S)^2$, $(J^M)^2$, and $(J^G)^2$ still have values $\frac{1}{2}$, M , and G respectively, but were omitted to avoid cumbersome notation. All quantum numbers are at this stage associated to rotationally invariant observables.

Now, observe that the binomial coefficient appearing in (11) is sharply peaked around the value $n = 0$, with a width $\Delta n \sim \sqrt{M}$. In this range, the term appearing under the square-root in (12) is one, plus fluctuations of order $1/\sqrt{M}$. Following guideline 2, we conclude that with probability approaching unity as M goes to infinity, the joint state of the system, magnet, and gyroscope is $|\Psi_n(t)\rangle \approx \alpha(t)|G + \frac{1}{2} + n; G + \frac{1}{2}\rangle^{SMG} + \beta(t)|G + \frac{1}{2} + n; G - \frac{1}{2}\rangle^{SMG}$ for some random $n \in [-\sqrt{M}, \sqrt{M}]$.

The *reduced* state of the system and the gyroscope—the state obtained by tracing out the relational degree of freedom involving the magnet—is given by

$$\begin{aligned} \rho_{\text{physical}}^{SG} \approx & |\alpha(t)|^2 |G + \frac{1}{2}; G; \frac{1}{2}\rangle \langle G + \frac{1}{2}; G; \frac{1}{2}| \\ & + |\beta(t)|^2 |G - \frac{1}{2}; G; \frac{1}{2}\rangle \langle G - \frac{1}{2}; G; \frac{1}{2}|, \end{aligned} \tag{13}$$

with interpretation that at time t , the system and gyroscope’s spin are either parallel or antiparallel with respective probabilities $|\alpha(t)|^2$ and $|\beta(t)|^2$. There are two ways to arrive at this result. One can start from (11), reverse the Clebsch–Gordan transformation and trace out the magnet. A more direct route is to use the fact that the map associated to tracing out a system and the map \mathcal{E} representing a group average as in (4) commute when the group acts unitarily on the system being traced out, i.e.,

$$\begin{aligned} Tr_B \int U^A(\Omega) \otimes U^B(\Omega) \rho^{AB} U^A(\Omega)^\dagger \otimes U^B(\Omega)^\dagger d\Omega \\ = \int U^A(\Omega) Tr_B \{ \rho^{AB} \} U^A(\Omega)^\dagger d\Omega. \end{aligned}$$

Thus, we can start directly from (8) and trace out the magnet. Up to corrections of order $1/\sqrt{M}$, the joint state of \mathcal{S} and \mathcal{G} will be given by (3) with time dependent amplitudes, so the results of Section 2.1 apply directly, yielding (13).

2.3. Time

So far, we have been concerned with the rotational symmetry of physical descriptions. The remaining symmetry is time translation. The explicit time parameter t appearing in the above equations is defined with respect to an unphysical reference frame, so must also be eliminated. But before getting rid of time, it is practical to build a clock! A clock \mathcal{C} is just a big rotating needle, so again it

will be represented quantum mechanically by a spin- C particle initialized in state $|C, C\rangle^C$. By letting this clock interact with a magnet \mathcal{N} (we use the letter \mathcal{N} for this magnet as \mathcal{M} is already used), it will start rotating just like a normal clock does. Of course, this clock has periodic motion—with period $T^C = \pi/\lambda N$ —so it can only keep track of time in a fixed interval $[0, T^C]$.⁴

In the real world, this problem is fixed by hooking up clocks to calendars, which break the periodicity. Here, we will circumvent this problem by assuming that the spacetime manifold (or sub-manifold of interest) has $t \in [0, T^C]$ with periodic boundary conditions, so our clock is well adapted. Here, periodic boundary conditions imply that as both magnets reach macroscopic size, their ratio $\Lambda = M/N$ is an integer, which is just saying that T^C is an integer multiple of the system’s precession period.

We will eliminate time using guideline 3, exactly as we did for the rotational reference frame: we perform a group average and remove the unphysical degrees of freedom. Given a time translation operator $U(t)$, we define the TPCP map \mathcal{T} by the action

$$\mathcal{T}(\rho) = \frac{1}{T^C} \int_0^{T^C} U(t)\rho U(t)^\dagger dt. \tag{14}$$

The combined effect of \mathcal{E} (cf. (4)) and \mathcal{T} is to randomize all nonphysical degrees of freedom, while keeping relational ones unchanged. Unphysical degrees of freedom can then be removed from the description.

To apply this procedure to our model, we use the the same tricks as above. Using (10), we obtain an expression of the time dependent state of the system, clock and two magnets. We can then trace out both magnets as both maps \mathcal{T} and \mathcal{E} act unitarily on them (in the case of \mathcal{T} , this is only true in the asymptotic limit). This yields the state

$$\begin{aligned} \rho^{SC}(t) \approx & \sum_{s,s'=\pm 1/2} \sum_{c,c'=-C}^C a_s a_{s'}^* \frac{1}{2^C} \sqrt{\binom{2C}{C+c} \binom{2C}{C+c'}} e^{it\{B(s-s')+B'(c-c')\}} \\ & \times |s\rangle\langle s'|_x^S \otimes |C, c\rangle\langle C, c'|_x^C \end{aligned} \tag{15}$$

where subscripts x indicate that the second quantum number refers to J_x , and $a_{\pm 1/2} = (\alpha \pm \beta)/\sqrt{2}$. Under the periodic boundary conditions of our model, the map \mathcal{T} will turn the exponential into a Kronecker delta $\delta_{c+\Lambda s, c'+\Lambda s'}$, where Λ is define above as the ratio of the two magnetic fields. This eliminates the time t from our physical description. We then introduce the gyroscope into the picture,

⁴In fact, since the gyroscope only allows us to read the clock along a single axis, the *observable* clock’s period is really $T^C/2$. More sophisticated clocks could be built, but we will ignore this problem by choosing Λ to be an even number for simplicity.

and express the state in terms of the operator $\sigma_z^S, (J^{CG})^2$ and J_z^{CG} . Using (6), we obtain

$$\begin{aligned} \rho^{SCG} \approx & \sum_c \sum_{s,s',r,r'} \frac{a_s a_{s'}^*}{2} \frac{1}{2^{2C}} \sqrt{\binom{2C}{C+c+\Lambda s} \binom{2C}{C+c+\Lambda s'}} \\ & \times (-1)^{(r-1/2)(s-1/2)+(r'-1/2)(s'-1/2)} |r\rangle\langle r'|^S \\ & \otimes \sum_{m,m'} d_{m,c+\Lambda s}^C d_{m',c+\Lambda s'}^C |G+m, G+m\rangle\langle G+m', G+m'|^{CG} \end{aligned}$$

where again, the quantum numbers for $(J^C)^2$ and $(J^G)^2$ are constant, so were omitted. The coefficients $d_{m,m'}^j = d_{m,m'}^j(\pi/2)$ are the Wigner rotation matrices (see e.g., (Sakurai, 1994)) that allow us to express J_x eigenstates in terms of J_z eigenstates. We can now apply the map \mathcal{E}^{SCG} to this state, and trace out the unphysical degrees of freedom to obtain

$$\begin{aligned} \rho_{\text{physical}}^{SCG} \approx & \sum_c \sum_{s,s',r,r'} \frac{a_s a_{s'}^*}{2} \frac{1}{2^{2C}} \sqrt{\binom{2C}{C+c+\Lambda s} \binom{2C}{C+c+\Lambda s'}} \quad (16) \\ & \times (-1)^{(r-1/2)(s-1/2)+(r'-1/2)(s'-1/2)} \\ & \times \sum_u d_{u-r,c+\Lambda s}^C d_{u-r',c+\Lambda s'}^C |G+u; G+u-r\rangle\langle G+u; G+u-r'|^{SCG} \end{aligned}$$

where the two quantum numbers a and b appearing in the states $|a; b\rangle$ refer to the relational observables $(J^{CG})^2$ and $(J^{SCG})^2$ respectively. This is the final result: it is the fully relational state obtained from applying the maps \mathcal{E} and \mathcal{T} and traced out both magnets from the initial non-relational state. We will now consider measurements statistics predicted by this state.

To “read the time,” one must measure the clock’s needle orientation relative to the gyroscope; in other words, measure $(J^{CG})^2$. This will fix the value of $G+u$, and hence of u . As before, this should be given the interpretation that the clock’s needle is at an angle θ satisfying $\cos \theta = u/C$. Just as with regular clocks, the angle of the needle θ is directly interpreted as time. We can evaluate the probability distribution $P(u)$ numerically from (16); we observe that the probability distribution for various values of u is roughly given by $1/\pi \sqrt{C^2 + u^2}$, see Fig. 1(a). This will lead to a *flat* distribution for the values of $\theta = \cos^{-1}(u/C)$ as expected. The interpretation of this result is that when reading the clock, one gets a random answer θ taking discrete values in $[0, \pi]$ with roughly equal probabilities.

Given a value of u , we can obtain the conditional state of the system, clock, and gyroscope by applying the von Neumann state update rule to $\rho_{\text{physical}}^{SCG}$ (von

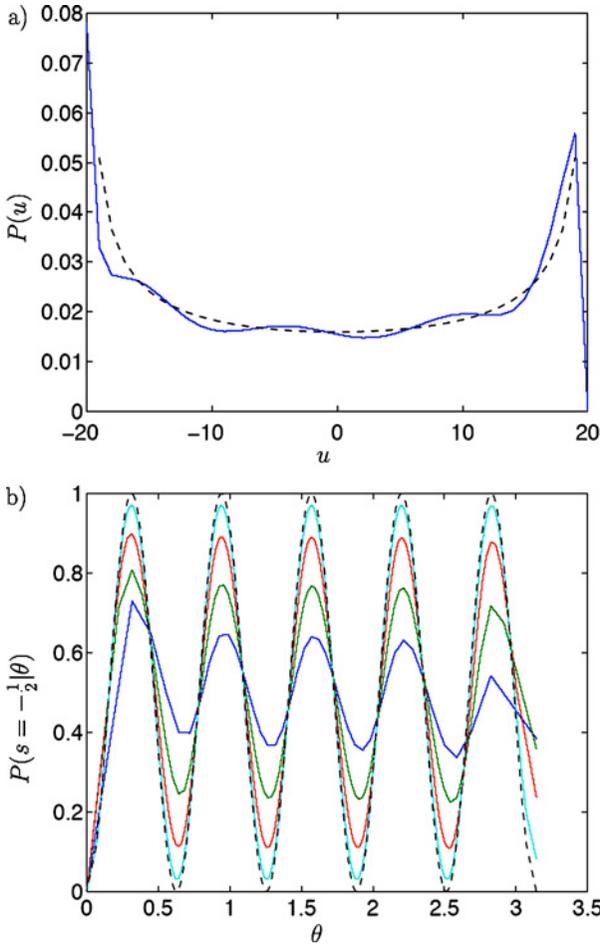


Fig. 1. Numerical results obtained from (17). The “free parameters” α and β , reflecting the system’s initial state in the nonrelational theory, were fixed to $\alpha = 1$ and $\beta = 0$. The ratio $\Lambda = M/N$ of the magnet’s size is fixed to 10. a) Probability of the outcome $G + u$ for the measurement of $(J^{CG})^2$ as a function of u , for a clock of size $C = 20$. The dash line is the function $1/\pi\sqrt{C^2 - u^2}$ corresponding to a flat distribution over the value of θ . b) Probability of the measurement outcome of $(J^{SCG})^2$ indicating an antiparallel orientation of the system and gyroscope’s spins as a function of clock reading θ , for various clock sizes $C = 20, 40, 100,$ and 400 . As the size of the clock increases, the result approaches the nonrelational prediction $P(r = +1/2|\theta) = |\beta(t)|^2$ given by (2), and illustrated by the dash line on the figure. The deterioration of the result as θ increases is due to a fundamental *decoherence* mechanism that occurs whenever clocks are of finite size (Gambini *et al.*, 2004).

Neumann, 1955):

$$\rho_{\text{physical}}^{SCG} \xrightarrow{u} \rho_{\text{physical}}^{SCG|u} = \frac{P_u \rho_{\text{physical}}^{SCG} P_u}{P(u)} \tag{17}$$

where P_u is the projector onto the subspace corresponding to the measurement outcome u . Since here the state $\rho_{\text{physical}}^{SCG}$ commutes with the various P_u , the state update rule is formally equivalent a classical Bayesian update, so it is compatible with guideline 4.

For a fixed value of clock read θ , we can ask what is the orientation of the system relative to the gyroscope and clock, or in other words, measure $(J^{SCG})^2$. (Note that measuring the orientation of the system relative to \mathcal{G} or $\mathcal{C} + \mathcal{G}$ gives the same result when \mathcal{G} is of macroscopic size.) This fixes the value of $G + u + s$, and hence of s . Starting from (16), we have numerically evaluated the probability of the outcome $s = -1/2$ —the system and gyroscope’s spin is antiparallel—given a value of θ for various gyroscope sizes; results are shown on Fig. 1. As expected, when the size of the clock is very large (e.g., $C = 400$) the results are very close to the orthodox predictions. For small clocks however, results agree for small values of θ and rapidly deteriorate. This is an interesting effect that we will discuss in Section 4.1. This completes our “translation” of the orthodox description of a system immersed into a magnetic field into a purely relational description.

3. SYMMETRIES AND NOISELESS SUBSYSTEMS

The techniques illustrated in the previous section were inspired by the noiseless subsystem method used to protect quantum states against undesired noise in quantum information science. The derivation we applied to the symmetry group $SO(3) \times U(1)$ representing rotation and time translation in the previous section can be applied straightforwardly to any symmetry group acting on a finite dimensional Hilbert space. The goal of this section is to present the general picture that naturally emerges from the four guidelines stated in Section 1.

Let us begin with a bit of notation. We consider a collection of quantum systems $\mathcal{S}_1, \mathcal{S}_2, \dots$ representing for example the system of interest, a clock, a gyroscope, etc., To avoid unnecessary mathematical complications, we will assume that each of these systems is associated a finite dimensional Hilbert space $\mathcal{H}^{\mathcal{S}_j}$, with $\dim(\mathcal{H}^{\mathcal{S}_j}) = d_j$. In a nonrelational theory, physical states are given by rays in the kinematical Hilbert space $\mathcal{H} = \bigotimes_j \mathcal{H}^{\mathcal{S}_j}$. These states can be expressed in terms of arbitrary basis $\{|e^1\rangle^{\mathcal{S}_j}, |e^2\rangle^{\mathcal{S}_j}, \dots, |e^{d_j}\rangle^{\mathcal{S}_j}\}$ which serve as a reference frame.

3.1. Noiseless Subsystems

A TPCP map $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ can be described in an operator-sum representation (Kraus, 1983) as $\mathcal{E}(\rho) = \sum_a E_a \rho E_a^\dagger$, with $\sum_a E_a^\dagger E_a = \mathbb{1}$ to ensure trace preservation. The algebra \mathcal{A} generated by the set $\{E_a, E_a^\dagger\}$ is a \dagger -algebra, called the *interaction algebra*, and as such it is unitarily equivalent to a direct sum of (possibly ‘‘ampliated’’) full matrix algebras: $\mathcal{A} \cong \bigoplus_J \mathcal{M}_{m_J} \otimes \mathbb{1}_{n_J}$, where \mathcal{M}_m is a m -dimensional full matrix algebra, and $\mathbb{1}_n$ is the $n \times n$ identity operator. This structure induces a natural decomposition of the Hilbert space

$$\mathcal{H} = \bigoplus_J \mathcal{H}_J \otimes \mathcal{K}_J,$$

where the ‘‘noisy subsystems’’ \mathcal{H}_J have dimension m_J and the ‘‘noiseless subsystems’’ \mathcal{K}_J have dimension n_J .

If \mathcal{E} is a *unital* quantum operation, by which we mean that the maximally mixed state $\mathbb{1}$ remains unaffected by \mathcal{E} (i.e., $\mathcal{E}(\mathbb{1}) = \mathbb{1}$), then the fundamental *noiseless subsystem* method (Knill *et al.*, 2000; Zanardi, 2001; Kempe *et al.*, 2001) of quantum error correction may be applied. This method makes use of the structure of the noise commutant,

$$\mathcal{A}' = \{ \rho \in \mathcal{B}(\mathcal{H}) : E\rho = \rho E \ \forall E \in \{E_a, E_a^\dagger\} \}, \tag{18}$$

to encode states that are immune to the errors of \mathcal{E} . Note that with the structure of \mathcal{A} given above, the noise commutant is unitarily equivalent to $\mathcal{A}' \cong \bigoplus_J \mathbb{1}_{m_J} \otimes \mathcal{M}_{n_J}$. Moreover, for unital \mathcal{E} , the noise commutant coincides with the fixed point set for \mathcal{E} (Busch and Singh, 1998; Lindblad, 1999); i.e.,

$$\mathcal{A}' = \text{Fix}(\mathcal{E}) = \{ \rho \in \mathcal{B}(\mathcal{H}) : \mathcal{E}(\rho) = \rho \}.$$

This means that a quantum state ρ will not be affected by the noise operation \mathcal{E} if and only if it is in \mathcal{A}' . We may thus regard the spaces \mathcal{K}_J as the Hilbert spaces associated to virtual particles that are not affected by the map \mathcal{E} . The noiseless subsystem technique has recently been generalized to include interaction algebras that are not \dagger -closed associated with nonunital maps (Kribs *et al.*, 2005; Nielsen and Poulin, 2005), but the unital case will be sufficient for our purposes.

3.2. Space Symmetries

We begin by considering ‘‘space’’ symmetries, i.e., those that are not time translation (we assume for a moment a nonrelativistic setting). The fundamental symmetries of the system are represented by a group \mathcal{G} which in general could be the Galilean group, the Weyl-Heisenberg group, etc. In the example of Section 2, this group was $SO(3)$. The discussion also straight forwardly extends to internal symmetries. The group \mathcal{G} acts unitarily on the state space of

each system: the effect of $g \in \mathcal{G}$ on the state $|\psi\rangle^{\mathcal{S}}$ of system \mathcal{S} is represented by some unitary matrix $U^{\mathcal{S}}(g)$. The kinematical state can be expressed in an arbitrary basis $\{|e^1\rangle^{\mathcal{S}}, |e^2\rangle^{\mathcal{S}}, \dots, |e^d\rangle^{\mathcal{S}}\}$ of the system. Given a group element $g \in \mathcal{G}$, we can construct a new set of basis states $\{|f^1\rangle^{\mathcal{S}}, |f^2\rangle^{\mathcal{S}}, \dots, |f^d\rangle^{\mathcal{S}}\}$ where $|f^k\rangle^{\mathcal{S}} = U^{\mathcal{S}}(g)|e^k\rangle^{\mathcal{S}}$ for all k .

From a quantum Bayesian point of view (Caves, et al., 2002; Fuchs, 2002), the nonrelational states should be thought of as states *given a preferred basis* or equivalently, *given a preferred reference frame* R . Thus, we should write $|\psi\rangle_{|R}^{\mathcal{S}_1\mathcal{S}_2\dots}$ for states expressed with respect to the reference frame R . Following the discussion of the previous paragraph, the same physical state can be expressed in terms of another reference frame R' as $|\psi\rangle_{|R'}^{\mathcal{S}_1\mathcal{S}_2\dots} = U^{\mathcal{S}_1}(g) \otimes U^{\mathcal{S}_2}(g) \otimes \dots |\psi\rangle_{|R}^{\mathcal{S}_1\mathcal{S}_2\dots}$, where g is the group element relating the basis associated to R and R' . But background independence tells us that R is unphysical. Following the Bayesian prescription, in the absence of an external reference frame, the state assigned to the collection of system should be a statistical mixture of the $|\psi\rangle_{|R}^{\mathcal{S}_1\mathcal{S}_2\dots}$ averaged over all reference frames. This leads to

$$\rho_{\text{physical}}^{\mathcal{S}_1\mathcal{S}_2\dots} = \mathcal{E}^{\mathcal{S}_1\mathcal{S}_2\dots}(|\psi\rangle\langle\psi|_{|R}^{\mathcal{S}_1\mathcal{S}_2\dots}) \tag{19}$$

$$= \int_{\mathcal{G}} U^{\mathcal{S}_1}(g) \otimes U^{\mathcal{S}_2}(g) \otimes \dots |\psi\rangle\langle\psi|_{|R}^{\mathcal{S}_1\mathcal{S}_2\dots} U^{\mathcal{S}_1}(g)^\dagger \otimes U^{\mathcal{S}_2}(g)^\dagger \otimes \dots dg \tag{20}$$

where dg is the group invariant measure satisfying $\int_{\mathcal{G}} dg = 1$. This choice of “flat” distribution reflects our complete ignorance of a preferred reference frame R , so it is well justified in a Bayesian approach. This defines a TPCP map $\mathcal{E}^{\mathcal{S}_1\mathcal{S}_2\dots}$ analogue to the one defined at (4) in our example of Section 2. This averaging procedure is the exact analog of the rule $P(a) = \sum_b P(a|b)P(b)$ of classical probability theory, relating the probability of event a to the conditional probability of a given a value of b and the prior probability of b .

This Bayesian inspired group average differs from the “coherent” group average $\int_{\mathcal{G}} U^{\mathcal{S}_1}(g) \otimes U^{\mathcal{S}_2}(g) \otimes \dots |\psi\rangle_{|R}^{\mathcal{S}_1\mathcal{S}_2\dots} dg$ commonly encountered in quantum gravity (see (Marolf, 2000) and references therein). In the case of rotational symmetry for example, the coherent group average simply projects onto the spin-zero subspace. This could annihilate the state, e.g., if the “Universe” contained an odd number of particles with half odd integer spin. A clear advantage of the “statistical” group average used here is that it is trace preserving. Thus, every kinematical state gets mapped to a normalized relational state by this procedure. Moreover, carrying the group average at the level of $\mathcal{B}(\mathcal{H})$ rather than \mathcal{H} may eliminate some mathematical complications that arise when the symmetry group only admits projective representations, i.e., when the left and right invariant Haar measure differ.

The map $\mathcal{E}^{\mathcal{S}_1 \mathcal{S}_2 \dots}$ constructed from tensor products of irreducible representations of the symmetry group induces a partition of the total Hilbert space

$$\mathcal{H} = \bigotimes_j \mathcal{H}^{\mathcal{S}_j} = \bigoplus_J \mathcal{H}_J \otimes \mathcal{K}_J \tag{21}$$

where J is a label for the different unitary irreducible representations of \mathcal{G} , \mathcal{H}_J is the sector on which the J th representation acts, and \mathcal{K}_J is the space associated to the degeneracy of the J th representation. By virtue of Schur’s lemma, the effect of the map $\mathcal{E}^{\mathcal{S}_1 \mathcal{S}_2 \dots}$ can easily be described in terms of this decomposition:

$$\mathcal{E}^{\mathcal{S}_1 \mathcal{S}_2 \dots}(\rho) = \sum_J \frac{\mathbb{1}_{\mathcal{H}_J}}{m_J} \otimes Tr_{\mathcal{H}_J} \{ P_J \rho P_J \} \tag{22}$$

for all $\rho \in \mathcal{B}(\mathcal{H})$. The projectors P_J are defined by $P_J \mathcal{H} = \mathcal{H}_J \otimes \mathcal{K}_J$, $\mathbb{1}_{\mathcal{H}_J}$ is the identity operator on \mathcal{H}_J , and $m_J = \dim(\mathcal{H}_J)$. The operation $Tr_{\mathcal{H}_J} : \mathcal{B}(\mathcal{H}_J \otimes \mathcal{K}_J) \rightarrow \mathcal{B}(\mathcal{K}_J)$ denotes the partial trace. In words, this map first imposes a superselection rule forbidding coherent superpositions across different J sectors. Then, within each superselected sector, it completely randomizes the state over the \mathcal{H}_J sector.

At this stage, the analogy with noiseless subsystems is clear. Losing an external reference frame induces some kind of “noise” into our physical description. We can think of each sectors \mathcal{K}_J as virtual *subsystems* that are immune to this noise. Obviously, these sectors must encode only *relational* information—information that is independent of any external reference frame. On the other hand, the sectors \mathcal{H}_J contain no information whatsoever about the physical system as they are always in a maximally mixed state in the absence of a reference frame. Thus, we can drop these “noisy” sector and simply write $\rho_{\text{physical}} = \bigoplus_J p_J \rho_J$ where the p_J were introduced so that the $\rho_J \in \mathcal{B}(\mathcal{K}_J)$ have unit trace. The interpretation of this state follows straightforwardly from guideline 4: the system is in one of the states ρ_J with respective probability p_J .

We note that, since each symmetry of the system is associated a conserved quantity, the superselection induced by the loss of an external reference frame implies superselection of conserved quantities. Hence, these quantities will always behave “classically,” as was noted before e.g., in the history formulation of quantum theory (Hartle *et al.*, 1995).

3.3. Time Symmetry

Time translational symmetry is treated in a completely analogous fashion. According to guideline 2, the Hamiltonian H has all the symmetries of the system; in other words, it should be time independent and invariant under the action of \mathcal{G} . A time-dependent quantum state should be thought of as a state *given an external*

clock C , and should accordingly be denoted $|\psi(t)\rangle_{|C}$. If C and C' are two clocks with associated time coordinate t and t' , we have $|\psi(t')\rangle_{|C'} = e^{iH(t-t')}|\psi(t)\rangle_{|C}$. Once again following the Bayesian prescription, the absence of such an external clock leads to time averaging

$$\rho_{\text{physical}} = \mathcal{T}(|\psi\rangle\langle\psi|_C) = \frac{1}{T} \int_0^T e^{-iHt} |\psi\rangle\langle\psi|_C e^{iHt} dt \quad (23)$$

where dt is the time translational invariant measure satisfying $\int_0^T dt = T$, and T is the period of the Hamiltonian H . The effect of \mathcal{T} will be to impose an energy superselection rule, so

$$[\rho_{\text{physical}}, H] = 0. \quad (24)$$

This may appear awkward since, in a nonrelational framework, this commutator generates the system's dynamics. But in a relational theory, the Hamiltonian naturally leads to a *constraint*. For example, the Wheeler-DeWitt equation (DeWitt, 1967) $H|\psi\rangle = 0$ is just a special case of (24).

Since H has the system's symmetries, it commutes with the elements of \mathcal{G} . Hence, the effect of the map \mathcal{T} will be to break up each of the sectors \mathcal{K}_J imposed by the geometrical symmetries into further noisy and noiseless sectors. One can use a more direct route and treat $\mathcal{G} \times U(1) = \{g \circ t : g \in \mathcal{G}, t \in [0, T]\}$ as the symmetry group \mathcal{G}' of the system and apply the noiseless subsystem techniques directly to the interaction algebra generated by \mathcal{G}' .

The end product of this procedure is a relational quantum state with one quantum number that is given the interpretation of time and that is *classically* correlated with the other quantum numbers. This so called *relational time* was first suggested by Page and Wootters (1983). In fact, these authors have described two distinct mechanisms by which dynamics could arise from a stationary state: either through quantum or classical correlations between the clock and the other degrees of freedom. When the state of the entire universe is *pure*, quantum correlations (or entanglement) are the only correlations available, so dynamics will unavoidably be caused by entanglement. This will be the case for example when considering the Wheeler-DeWitt equation. This type of clock has been investigated by many (Unruh and Wald, 1989; Pegg, 1991; Gambini and Porto, 2001). However, Eq. (24) admits mixed state solutions, and we arrived at mixed state description following an arguably reasonable set of guidelines (particularly from a Bayesian perspective) and plain quantum mechanics. This might be a hint that time arises through *classical* correlations rather than entanglement, and that the Hamiltonian constraint equation should be imposed at the level of $\mathcal{B}(\mathcal{H})$ rather than \mathcal{H} . These distinctions between our construction and what is customarily assumed can be traced back to the use of a statistical rather than a coherent group average.

Finally, we note that the observables used to arrive at a non-trivial time evolution which mimics the orthodox predictions—as in Fig. 1(b)—are not observables in Dirac’s sense for constraint systems: they do not commute with the Hamiltonian. This is an unavoidable drawback of the relational time paradigm. In the presence of interactions, the universe cannot generally be divided into subsystems without breaking time reparameterization symmetry. In the toy model, time symmetry is broken when the magnets are traced out of the physical description.

4. DISCUSSION

4.1. Fundamental Decoherence

We see on Fig. 1(b) that the relational theory does not reproduce the orthodox predictions exactly. In fact, the prediction become worst as the clock time parameter θ increases: curves are closer to the dash curve at $\theta = 0$ than they are at $\theta = \pi$. Moreover, these effect decrease as the size of the clock increases, but are always present for finite clocks. This is due to the fact that time is now a quantum variable, and as such it is subject to quantum fluctuations. The effect of such a diffuse time has been investigated as a fundamental decoherence mechanism (Gambini *et al.*, 2004, 2005; Milburn, 2003; Milburn and Poulin, 2005) taking pure states into mixed states (Zurek, 2003).

Again, the Bayesian approach helps understanding the origin of this fundamental decoherence. In the absence of quantum fluctuation of the clock variable, the state of the systems conditioned on the clock reading t is given by $\rho(t) = e^{-iHt} \rho(0) e^{iHt}$. However, the reading of a finite dimensional clock yields a diffuse time value: time is known within a finite accuracy. Thus, given a clock reading “ θ ,” the state of the system should be

$$\rho(\theta) = \int P(t|\theta) e^{-iHt} \rho(0) e^{iHt} dt \tag{25}$$

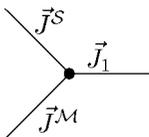
where $P(t|\theta)$ represents our *a posteriori* probability distribution over the value of t given our knowledge of θ . The decoherence rate will be directly related to the width of this distribution (Milburn, 2003; Milburn and Poulin, 2005), which in turn is a function of the clock’s size. Determining optimal tradeoff between clock-size and decoherence rate in a relational theory is obviously an interesting question and we leave it for future investigation. Moreover, the finite-size effect of the other systems \mathcal{G} , \mathcal{M} , and \mathcal{N} —which we have assumed to be infinite in our numerical analysis—will add on to this disagreement between orthodox and relational predictions. For example, Eq. (5) leads to slightly modified probability rule when the gyroscope is of finite size.

It is interesting to note that this effect introduces an arrow of time: even though the relational theory is time-symmetric (or more precisely time-independent), it

yields predictions that are not symmetric with respect to the clock time, as is clearly illustrated on Fig. 1(b). This suggests that the arrow of time could emerge from the finite size of our clocks, and that this could be verified experimentally. Recall however that our setting only allows us to measure the clock’s spin along a single axis, and as a consequence we cannot distinguish the time range $\theta \in [0, \pi]$ from $\theta \in [\pi, 2\pi]$. Being able to distinguish the time range $\theta \in [\pi, 2\pi]$, e.g., by introducing a second gyroscope aligned along the y axis, we would observe a “recoherence” phase during the second half of the universe’s period. Indeed, this follows from the periodic boundary conditions: the amplitude of the oscillation at time $\theta = 2\pi$ is equal to the amplitude at time $\theta = 0$. Such recoherence can only occur if $\partial\rho^{S\mathcal{G}}/\partial\theta$ —the derivative of the system’s state with respect to the clock time θ —is nonlocal in time; if the equation of motion has a memory term. In a Markovian approximation where we neglect the memory term, the recoherence phase will disappear, leading to an effective arrow of time. A detailed study of this effect is left to future investigation.

4.2. Spin Networks

We will now revisit our toy model and introduce a diagrammatic representation for every step that went into our calculations. The five quantum systems \mathcal{S} , \mathcal{M} , \mathcal{C} , \mathcal{N} and \mathcal{G} are each assigned an angular momentum operator $\vec{J}^{\mathcal{S}}$, $\vec{J}^{\mathcal{M}}$, $\vec{J}^{\mathcal{C}}$, $\vec{J}^{\mathcal{N}}$, and $\vec{J}^{\mathcal{G}}$ respectively. The first step to solve the nonrelational dynamical equations was to express the state of \mathcal{S} and \mathcal{M} in terms of their total angular momentum operator, and similarly for the pair $\mathcal{C} - \mathcal{N}$. Thus, we define two new operators \vec{J}_1 and \vec{J}_2 satisfying $\vec{J}^{\mathcal{S}} + \vec{J}^{\mathcal{M}} + \vec{J}_1 = 0$ and $\vec{J}^{\mathcal{C}} + \vec{J}^{\mathcal{N}} - \vec{J}_2 = 0$ (the signs might appear arbitrary but are necessary). We can represent this graphically as follows:



and similarly for the $-\vec{J}^{\mathcal{C}}, -\vec{J}^{\mathcal{N}}, \vec{J}_2$ triplet.

In the next step, we combined the angular momentum of the gyroscope to \vec{J}_2 in order to “read the time.” This defines a new operator \vec{J}_3 satisfying the relation $\vec{J}^{\mathcal{G}} + \vec{J}_2 + \vec{J}_3 = 0$, or in other words $\vec{J}_3 = -(\vec{J}^{\mathcal{C}} + \vec{J}^{\mathcal{N}} + \vec{J}^{\mathcal{G}})$. Finally, to measure the system’s state relative to the gyroscope and clock, we have defined yet another operator \vec{J}_{total} satisfying the relation $\vec{J}_1 + \vec{J}_3 + \vec{J}_{\text{total}} = 0$, or equivalently $\vec{J}_{\text{total}} = \vec{J}^{\mathcal{S}} + \vec{J}^{\mathcal{M}} + \vec{J}^{\mathcal{C}} + \vec{J}^{\mathcal{N}} + \vec{J}^{\mathcal{G}}$. Combining all these steps together yields the graph

The decorated graphs Γ —called *spin networks* (Penrose, 1971; Rovelli and Smolin, 1995; Baez, 1995; Major, 1999)—corresponds to basis states $|\Gamma\rangle$ for the relational theory’s Hilbert space \mathcal{H}_{rel} . In our model, the free edges must have a fixed j value, or a statistical mixture of such values, while the interior edges can be in quantum superpositions. A vertex with incoming edges labeled j_1 , j_2 , and j_3 is associated an *intertwining operator*, that is a map $\mathbb{C}^{2j_1+1} \otimes \mathbb{C}^{2j_2+1} \otimes \mathbb{C}^{2j_3+1} \rightarrow \mathbb{C}$. In the case of trivalent graphs, this map simply gives the Clebsch–Gordan coefficients needed to go from (27) to (28). The map \mathcal{T} is a “sum over histories” of the graphs, e.g., $\mathcal{T}(\rho) = \int U(t)\rho U(t)^\dagger dt$. In the loop approach to quantum gravity, this sum is performed using *spin foams* (see e.g., Baez, 2000) which are analog to Feynman diagrams used in quantum electrodynamics. However, the sum over histories is here carried at the level of $\mathcal{B}(\mathcal{H}_{\text{rel}})$, not on \mathcal{H}_{rel} as it is usually the case; it is not a coherent sum. Again, this is a consequence of the Hamiltonian constraint $[\rho, H] = 0$ that naturally arises in our theory and differs from the usual equation $H|\psi\rangle = 0$.

The fact that spin networks can serve as basis states in our relational model is not very surprising since they can be used to describe gauge independent observables in Yang–Mills theories, e.g., generalized Wilson loop operators (Kogut and Susskind, 1975; Baez, 1996). Nevertheless, we believe that there is a lot to be learned about the low energy limit of spin foams models of quantum gravity from this simple analogy. In the absence of experimental guidance, connections with well-established physical regimes of the theory can be a quite useful. Spin networks are likely to play an important role in quantum gravity, but their low energy limit is poorly understood. Deriving them from textbook quantum mechanics combined with arguably reasonable guidelines can thus yield interesting insights. One could of course attempt to repeat the construction with a different (and undoubtedly more interesting) symmetry group with the help of the studies pursued in the context of spin networks (see e.g., Freidel and Livine, 2003), but this is beyond the scope of this paper.

On the other hand, our construction also shows how the tools developed in quantum gravity can be useful in quantum information science. In particular, we can consider the group \mathcal{G} generated by the interaction algebra \mathcal{A} of a collective TPCP map \mathcal{E} . Then, we can construct spin networks decorated by irreducible representation of \mathcal{G} , just like we did here for the group $SU(2)$. The state space of the *noiseless subsystems* will thus correspond to the inter-twiner space of the spin network (Girelli and Livine, 2005).

4.3. Generalizations and Connections

The basic idea exploited in the present paper is to eliminate nonrelational degrees of freedom by performing averages over the symmetry group of the system. This technique is well justified from a Bayesian point of view and relies on the

tools of quantum information science. The idea extends well beyond the example presented here: it provides a general framework for the study of constrained quantum systems. Following a similar line of thoughts, Aharonov and Susskind (1967) have demonstrated how constraints arising from superselection rules can be effectively circumvented in the presence of a quantum reference system. In the context of charge superselection rule, their construction leads to the possibility of observing interference between states of different charges. The question of a quantum reference of phase has also been debated in the laser (Mølmer, 1997), the Bose-Einstein condensate (Leggett, 2000), and the superconductor (Kershaw and Woo, 1974) communities. In all these cases, the emerging (not yet unanimous) consensus is that the existence or non-existence of a phase depends on whether the reference system is treated classically or quantum mechanically (Bartlett, Rudolph, and Spekkens, 2004). As we have shown here, these two descriptions are equivalent in appropriate regimes.

Our method can also be used to argue that spontaneous symmetry breaking—an important concept in condensed matter and high-energy physics related to the existence of Goldstone bosons (Goldstone, Salam, and Weinberg, 1962)—is not a fundamental phenomenon. Goldstone modes can be observed without symmetry breaking in the presence of a proper system of reference. For example, the ground state of a Heisenberg ferromagnet can single out a polarization *relative* to a system of reference while leaving the global state of the ferromagnet and reference system rotationally invariant. Thus, while symmetry-broken states are useful to describe thermodynamic systems, an equivalent description exists that preserves all symmetries of nature.

As mentioned at the end of the last section, the next logical step is to apply the technique to more interesting symmetry groups. This program should be tractable for the Lorentz group of a free field (Toller, 1997; Mazzucchi, 2000) since quantum field theory is already ideally set up for this purpose, with the different particles corresponding to different irreducible representations of the symmetry group. The noncompactness of the Lorentz group certainly create extra mathematical complications, but can nonetheless be handled in principle (Freidel and Livine, 2003). A system of harmonic oscillators also has a noncompact symmetry group, yet our program applies almost straightforwardly to this case (Milburn and Poulin, 2005). For an interacting field however, the group average becomes very difficult to perform; one can instead try to formulate the theory directly in terms of invariant observables (as in algebraic quantum field theory Haag, 1992).

One should also attempt to extend the program to local symmetries, i.e., gauge groups. A lattice Abelian gauge theory would be a good place to start. In this context, gauge fixing is equivalent to introducing a nonphysical “reference frame” for the gauge field. Hence, our construction would eliminate the gauge degree of freedom, and the emerging relational quantum theory should not

contain any gauge field; it should be expressed entirely in terms of Wilson loop like observables—hence with spin networks (Kogut and Susskind, 1975; Baez, 1996).

5. CONCLUSION

In this paper, we have argued that when described at a fundamental level, i.e., in the absence of semiclassical approximations, quantum theory is naturally relational. Quantum states are independent of any external reference frame or clock, but only describe relations between physical systems. A nonrelational description can be recovered as a semiclassical approximation of the relational theory. We have illustrated this thesis with the help of a collection of particles with spin—which we use to model a clock, a gyroscope, magnets, etc.—and discussed how the idea carries over to general quantum systems. Our construction was motivated by a Bayesian approach to quantum mechanics and borrowed tools from quantum information science. Using elementary quantum mechanics, we have re-derived some well-known concepts. By treating time as a quantum mechanical system—a clock—we were naturally lead to the notion of relational time of Page and Wootters (1983). Quantum fluctuations of the clock variable predicts a fundamental decoherence mechanism, recently discussed by Gambini, Porto, and Pullin (2004), and clearly seen in our numerical analysis. As a consequence, an effective arrow of time emerges from a time-independent theory. Finally, basis states for the relational theory can be described in terms of spin-network introduced by Penrose (1971) and extensively studied in loop quantum gravity (Baez, 1995; Rovelli and Smolin, 1995).

However, these concepts emerged from our construction in a slightly different way than they usually do, and this might lead to interesting physical insights. The most important distinction concerns the Hamiltonian constraint: we found that physical states must commute with the Hamiltonian, while it is usually assumed that physical states must be annihilated by the Hamiltonian. This distinction is very important, specially when considering mixed states solution that are naturally expected in a Bayesian approach. As a consequence, we found that relational time rises from *classical* correlations between the clock and system of interest, not entanglement as it is usually assumed. This distinction also suggests that the sum over histories associated to the Hamiltonian constraint—implemented with spin foams in loop quantum gravity—should be carried at the level of *operators* rather than vectors of the Hilbert space. It is an open question whether this statistical group average can be consistently and formally incorporated into a generally covariant theory, and if so, whether it leads to physical predictions that differ from those obtained through coherent group average. Since quantum gravity is difficult to probe experimentally, deriving some of its concepts from familiar and

overwhelmingly tested regimes of quantum theory as we did in this paper may offer valuable physical guidance.

Finally, our study illustrates the usefulness of quantum information science—and particularly the Bayesian view—in quantum gravity, Wheeler’s “it from bit” (Wheeler, 1991); a point of view that has been gaining popularity lately, e.g. (Girelli and Livine, 2005; Lloyd, 2005). Conversely, we have also exposed how the tools developed in a background independent theory such as loop quantum gravity have direct applications in quantum information: e.g., by demonstrating that the intertwiners form a basis for the noiseless subsystem of a collective noise operation, all the mathematical baggage of spin networks carries over to the study of noiseless subsystems.

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