Universal Topological Phase of 2D Stabilizer Codes

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outline

• motivation
• anyons & toric code
• universality of toric code
• subsystem codes
• conclusions
motivation

- quantum error correcting codes fight decoherence with (subtle) redundancy
- typically this involves encoding in a **subspace**

![Hilbert space and Code Subspace diagram]
• the code subspace can be defined in terms of commuting observables: **check operators** (CO)

\[ C_i |\psi\rangle = c_i |\psi\rangle \]

• errors typically change CO values
• this allows to keep track of errors:

**CO measurement \(\rightarrow\) error syndrome \(\rightarrow\)

\(\rightarrow\) compute most probable error
• conventional quantum fault-tolerance can be unpractical
• when locality matters: topological codes (Kitaev ‘97)

- error threshold: perfect storage in the large size limit
• closely related: **topological order** (TO) in condensed matter physics (Wen ‘89)
  – gapped excitations
  – topology dependent ground state (GS) degeneracy
  – different GSs are locally identical
  – robust under local perturbations (Bravyi et al ’10)
  – cannot be classified through local symmetries

*topological order describes the equivalent classes defined by local unitary evolutions*

(Chen, Gu, Wen ’10)
they argue that **two gapped GSs**...

...are in the same phase

\[\leftrightarrow\]

...can be connected adiabatically without closing the gap

\[\leftrightarrow\]

...are connected by a local unitary

\[\leftrightarrow\]

...are connected by a quantum circuit of constant depth
• they define **short- and long-range entanglement**:

A state has only short-range entanglement if and only if it can be transformed into an unentangled state (i.e. a direct-product state) through a local unitary evolution.

• in this sense, TO is a pattern of long-range entanglement

classify topological order = classify entanglement
(pure states up to local unitaries)
• **topological stabilizer codes** (TSC) provide a wide but manageable class of topological codes/models

\[
H = - \sum_{s \in S_g} s, \quad S_g \subset P := \langle i1, X_k, Z_k \rangle
\]

• stabilizer group:

\[
S = \langle S_g \rangle, \quad -1 \notin S
\]

• ground/code subspace:

\[
S |\psi\rangle = |\psi\rangle
\]
• we consider 2D TSCs that are translationally invariant

1\textsuperscript{st} goal: show rigorously that local equivalence works

2\textsuperscript{nd} goal: find general structure of 2D TSCs (including subsystem codes)

• among other things, this allows to extend results and techniques well known in specific TSCs, e.g. error correction in toric codes (Dennis et al ‘02, Duclos-Cianci & Poulin ‘10, ...)
anyons & toric code

• 2D systems are especial regarding particle statistics

• anyons: topological interactions
• abelian anyon models:

- Topological charges:
  \[ q \in \{a, b, \ldots \} \]

- Fusion rules:
  \[ q_1 \times q_2 = q \]

- Braiding rules:
  \[ |\psi\rangle \rightarrow e^{i\alpha_{12}} |\psi'\rangle \]
• example: **toric code** (Kitaev ’97):
  • qubits at the edges of a square lattice
  • 4-local Pauli interaction terms at faces (direct & dual)
  \[ H = - \sum_f S_f - \sum_{f^*} S_{f^*} \]
  • ground state: \( S_f = S_{f^*} = 1 \)
  • excitations live at dual and direct faces
• **string operators** create/annihilate excitations at their endpoints

• two types of strings/excitations: e (dual faces) and m (direct faces)

• excitations can be created/annihilated in pairs only

• there are four charges: \(1, e, m, \epsilon = m \times e\)
• crossing string ops with different charge anticommute

\[ pq = -qp \]

• this gives rise to semionic topological interactions

\[ p^{-1}q^{-1}pq = -p^{-1}pq^{-1}q = -1 \]
• self-statistics also follow from string commutation relations

• e and m are bosons, the composed charge is a fermion

\[ q^{-1} r p^{-1} q r^{-1} p = -q^{-1} q p^{-1} r r^{-1} p = -1 \]
• abstract **anyon model**

• charges: \( \{1, e, m, \epsilon\} \)

• fusion:

\[
\begin{align*}
e \times m &= \epsilon \\
e \times \epsilon &= m \\
m \times \epsilon &= e \\
e \times e &= m \times m = \epsilon \times \epsilon &= 1
\end{align*}
\]

• braiding: \( e, m \rightarrow \text{bosons} \quad \epsilon \rightarrow \text{fermion} \)
• topological ground state degeneracy
• torus: 4-fold degenerate
• closed non-trivial strings provide **logical operators**
universality of toric code

• TSCs are usually given as a recipe to construct check ops (Hamiltonian terms) on a given lattice of qubits
• assume that the bulk of the code is local and translationally invariant (LTI)
• **lattice stabilizer group** (LSG): apply the LTI recipe to an infinite version of the lattice
• we are only interested in LSGs coming from TSCs where

local undetectable errors do not affect encoded states

• but elements of $\mathcal{Z}_P(S)$ map back to such errors

• thus, define **topological stabilizer groups** (TSG) by

$$\mathcal{Z}_P(S) \propto S$$
• goal: classify TSGs up to local unitaries
• it turns out to be enough to consider **equivalence** up to the following LTI operations

- coarse graining
- Pauli isomorphisms
- add/remove disentangled qubits
• TSCs with the same anyon model are equivalent!
• we rule out chiral anyons, since the Hamiltonian terms commute with each other (Kitaev ‘06)

**every 2D TSC is locally equivalent to a finite number of copies of KTC**

• equivalence classes labeled by the total quantum dimension or the topological entanglement entropy
• outline of the proof:

1. TSGs admit LTI independent generators
2. there is a finite # global constraints
3. # global constraints = # charges
4. charges are topological: string operators
5. ruling out chirality, multiple copies of KTC anyons
6. framework of string segments as in multiple KTCs
7. other stabilizers (non-plaquettes) have no charge
8. map: string segments <---> string segments, uncharged stabilizers <---> single qubit stabilizers
1. TSGs admit LTI independent generators

- the centralizer of a 2D LPG is a LPG and admits LTI independent generators
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![Diagram](image-url)
1. **TSGs admit LTI independent generators**

- the **centralizer** of a 2D LPG is a LPG and admits LTI independent generators
3. \# global constraints = \# charges

- trick: construct independent generators of global constraints from independent generators of charges
- stabilizer generators have a charge
- constraints only contain complete sets of stabilizers with a given charge
3. \# global constraints = \# charges

- for each generating charge $c$, take all stabilizers with charges that have $c$ as a component
- no Pauli op \textit{anticommutes} with the product, or it would create net charge $c$, so it is a global constraint
4. **charges are topological: string operators**

- after coarse graining, uncharged excitation sets can be removed with **string-like operators**
4. charges are topological: string operators

- extract an **anyon model** from the commutation rules:

\[
\kappa(\bullet, \bullet) = (p; q) := pqp^{-1}q^{-1}
\]

\[
\theta(\bullet) = (p; q)(p; r)(q; r) = (p; q)(p; r)
\]
4. *charges are topological: string operators*

- well defined, independent of the string choice:

\[(p; q) ?= (p'; q')\]
4. *charges are topological: string operators*

- well defined, independent of the string **choice:**

\[(p; q) = (p'; q')\]

\[\in S\]
6. framework of string segments as in multiple KTCs

- commutation of string segments can be made **model independent** thanks to the topological charge they carry
8. map: string segments $\leftrightarrow$ string segments, uncharged stabilizers $\leftrightarrow$ single qubit stabilizers
subsystem codes

- more general encoding strategy (Kribs et al ’05)
• check ops measured indirectly → potentially more local (Poulin ‘05)

• **topological subsystem color codes** (Bombin ‘10)
  – 2-local measurements
  – string ops: chiral anyons, 3 fermions with semionic interactions
  – all Clifford gates by code deformation
• in general, subsystem stabilizer codes are defined by a **gauge group** (its elements do not affect encoded states)

\[ G \subset P \quad \mathbb{Z}_g(G) \propto S \]

• from the recipe for a subsystem TSC we can construct a subsystem TSG, defined by the **topological condition**

\[ \mathbb{Z}_P(\mathbb{Z}_P(S)) \propto S \]

• define

\[ G := \mathbb{Z}_P(S) \]
• for a TSG $\mathcal{A}$, **charges** are the elements of the quotient

$$C_A := \frac{\Phi^0(\mathcal{A})}{\text{Com}_A(\mathcal{P})}$$

**charge group** := morphisms from $\mathcal{A}$ to $\pm1$

commutators with Pauli ops

• but here $\mathcal{A}$ could be also a gauge group!
• gauge charges map naturally to stabilizer charges

\[ C_i : C_G \rightarrow C_S \]

• in general not an isomorphism, but still

\[ |C_S| = |C_G| \]

• indeed, these charge groups are naturally dual!

\[ C_S \in \mathcal{G} \]
\[ C_G \in \mathcal{S} \]

\[ \kappa : C_G \times C_G \rightarrow \pm 1, \]
\[ \theta : C_G \rightarrow \pm 1, \]
\[ \kappa : C_G \times C_S \rightarrow \pm 1, \]
• only gauge charges give an anyon model
• we have
  \[ \kappa(c, d) = \kappa(c, C_l(d)) \quad c, d \in C_G \]
• we can find generators
  \[ C_G = \langle c_1, \ldots, c_\alpha, d_1, \ldots, d_\alpha, e_1, \ldots, e_\beta \rangle \]
  \[ C_S = \langle C_l(c_1), \ldots, C_l(c_\alpha), C_l(d_1), \ldots, C_l(d_\alpha), \tilde{e}_1, \ldots, \tilde{e}_\beta \rangle \]
• all mutual statistics for generators are trivial except
  \[ \kappa(c_i, d_i) = -1 \quad \kappa(e_j, \tilde{e}_j) = -1 \]
• all gauge charge generators with index \( \neq 1 \) are bosons
all possible anyon models are combinations of
- toric code: 2 bosons, 1 fermion, semionic interactions
- TSCC: 3 fermions, semionic interactions (chiral)
- subsystem toric code: 1 boson
- honeycomb subsystem code: 1 fermion

a construction with string segment ops gives a standard form of the codes

**every 2D TSC has a structure dictated by an anyon model**
• it is possible to go back to a finite periodic lattice
• logical ops: homologically nontrivial string ops
• each non-anyonic charge requires a global stabilizer or gauge generator

\[ C_S = \langle \bar{c}, \bar{d}, \bar{e} \rangle \]
\[ C_G = \langle c, 1, e \rangle \]
\[ \{ T(s) \} \in \mathbb{Z}_{\leq S} \]
\[ \ell \in G \]
conclusions & questions

• the long-range entanglement pattern of toric codes is universal for 2D topological stabilizer models
• all 2D TSCs are anyonic
  • the same approach could be used for boundaries or point defects...
• more general 2D models?
• what about 3D?