# Stability of Frustration-Free Hamiltonians

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Joint work with Justyna Pytel - Oregon State University.

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# The challenge...

``Find a minimal set of assumptions under which gapped Hamiltonians have stable spectral gap against weak, local perturbations.''

**Isn’t every gapped system stable?**

# Counterexamples to stability: Opening the gap.

Splitting the groundstate degeneracy.

## Example

1. Consider the $N \times N$ Ising Hamiltonian $H_N$ and a perturbation $\Delta_N$:

$$H_N = -\sum_{p\sim q} \sigma_p^z \otimes \sigma_q^z, \quad \Delta_N = \delta_N \sum_p \sigma_p^z, \quad \delta_N \sim 1/N^2.$$

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4. Good classical memory, bad quantum memory: The state $|+\rangle = |000\ldots 0\rangle + |111\ldots 1\rangle$ flips to $|-\rangle = |000\ldots 0\rangle - |111\ldots 1\rangle$, since $e^{itH_N'}|+\rangle \sim |000\ldots 0\rangle + e^{it}|111\ldots 1\rangle$.

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4. $H_N' = H_N + \delta_N \sum_p \sigma_p^z$ has **degenerate groundstate** space spanned by $|000 \cdots 0\rangle$ and $|111 \cdots 1\rangle$, for **vanishing** $\delta_N \sim 1/N^2$.

# Distinguishability implies instability!

Hamiltonians are **unstable** because **local order parameters** can act as perturbations to **split the groundstate subspace**, or **close the gap** between **groundstates** and **local, low-energy excitations**.

# Frustration-free Hamiltonians.

Definition

1. We say $\mathbf{H_0} = \sum_{\mathbf{u}\in\mathbf{\Lambda}} \mathbf{Q_u}$ is **frustration-free**, if the groundstate subspace $P_0$ satisfies for all $u \in \Lambda \subset \mathbb{Z}^d$:

$$Q_u P_0 = \lambda_u P_0,$$

where $\lambda_u$ is the **smallest eigenvalue** of $Q_u$. From now on, we will substitute $Q_u$ with $Q_u - \lambda_u \mathbb{1} \geq 0$. This generates a global energy shift, which is irrelevant for the spectral gap.

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2. NOT all COMMUTING Hamiltonians are FRUSTRATION-FREE!
Take 3 qubits on the vertices $\{u, v, w\}$ of a triangle, with Ising Hamiltonian $H_\triangle = \sigma_u^z \otimes \sigma_v^z + \sigma_v^z \otimes \sigma_w^z + \sigma_u^z \otimes \sigma_w^z$.
Since $\sigma_u^z \otimes \sigma_w^z = (\sigma_u^z \otimes \sigma_v^z) \cdot (\sigma_v^z \otimes \sigma_w^z)$, it is impossible to have a common eigenvector with eigenvalue $-1$, for all three terms.

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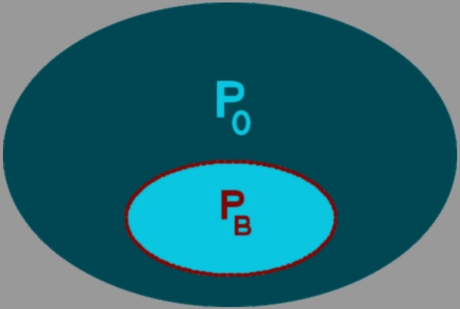
3. NOT all FRUSTRATION-FREE Hamiltonians are COMMUTING!
   Take any generic parent Hamiltonian of a Matrix Product State.

# Local Groundstates




Every frustration-free Hamiltonian $H_0$ on $\Lambda$ is the extension of another frustration-free Hamiltonian $H_B$ on $B \subset \Lambda$. This implies that the local groundstate projector $P_B$ contains $P_0$; that is, $P_B P_0 = P_0$.

# Topological Quantum Order

Macroscopic indistinguishability of global groundstates.

**TQO: $\mathbf{P_0}$** satisfies **Topological Quantum Order**, if for all $O_A$:

$$P_0 O_A P_0 = c(O_A) P_0, \quad c(O_A) = \mathrm{Tr}(O_A\, P_0)/\,\mathrm{Tr}\, P_0, \quad (1)$$

where $A = b_u(r)$, $\mathbf{r} \leq \mathbf{L}^* \sim \mathbf{L}^\alpha$, $\alpha \in (0, 1]$.

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Some simple examples.

1. The Ising Hamiltonian from Counterexample I, does not satisfy the *TQO* condition, since the groundstate subspace is spanned by the product states $|000\cdots 0\rangle$ and $|111\cdots 1\rangle$.

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2. In particular, $P_0 O_A P_0 = c(O_A) P_0$ implies that $\langle\psi_0| O_A |\psi_0\rangle = \langle\phi_0| O_A |\phi_0\rangle$ for any two groundstates $|\psi_0\rangle, |\phi_0\rangle$. But, $\langle 000\cdots 0| \sigma_0^z |000\cdots 0\rangle = 1$ and $\langle 111\cdots 1| \sigma_0^z |111\cdots 1\rangle = -1$, so *TQO* fails in this case.

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3. It is no coincidence that the field $\sum_p \sigma_p^z$ was used to split the groundstate subspace of the Ising Hamiltonian.
4. Kitaev's Toric Code, a four-fold degenerate groundstate subspace, satisfies the *TQO* condition with $\alpha = 1$, so $L^* \sim L$.

# Kitaev's Toric Code

The toric code model

Example

The standard toric code model is defined by the Hamiltonian:

$$H_{tc} = -\sum_p B_p - \sum_s A_s,$$

where qubits live on the edges of a lattice on a **torus**.

- Lowest-energy subspace $P_0$ (toric code) has $B_p = 1$, $A_s = 1$ for all $p$ and $s$. That is, for any ground state $|\Psi_0\rangle$ we have:

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- Since $\prod_s A_s = \prod_p B_p = \mathbb{1}$, there are 4 such ground states on the torus, distinguished only through **macroscopic string operators**.

# Breaking the Toric Code

The Ising code model

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Assume we are given the following Hamiltonian:

$$H_{ic} = -\sum_{p\sim p'} B_p \otimes B_{p'} - B_0 - \sum_s A_s,$$

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- Perturbing the Hamiltonian by adding the vanishing **B-field** $\delta \sum_p B_p$, with $\delta \sim 1/|\Lambda|$, we close the gap!
- The new sector $A_s = 1, B_p = -1$, becomes a groundstate subspace. So, **TQO** is not enough for stability.

# Stability needs...

Local Groundstate Indistinguishability.

**Local-TQO:** $\mathbf{H_0}$ satisfies **Local-TQO**, if there exists a **rapidly-decaying function** $\Delta_0(\ell)$, such that:

$$\|P_{A(\ell)} O_A P_{A(\ell)} - c(O_A) P_{A(\ell)}\| \leq \|O_A\| \, \Delta_0(\ell). \qquad (3)$$

Here, $A = b_u(r)$, $r \leq L^*$ and $A(\ell) := b_u(r+\ell)$.

**Note:** The above condition is equivalent to assuming that groundstates on $A(\ell)$ are identical when viewed on $A$, up to error $\Delta_0(\ell)$:

$$\|\rho_A^1 - \rho_A^2\|_1 = \sup_{\|O_A\|=1} \left| \left\langle \psi^1_{A(\ell)} \right| O_A \left| \psi^1_{A(\ell)} \right\rangle - \left\langle \psi^2_{A(\ell)} \right| O_A \left| \psi^2_{A(\ell)} \right\rangle \right| \leq 2\Delta_0(\ell).$$

# Local-TQO implies Area Law.

Corollary

*Let $A = b_u(r)$, with $r \le L^*$ and $u \in \Lambda$. Any groundstate $|\Psi_0\rangle$ of $H_0$ satisfying Local-TQO, also satisfies an* **area law** *for the entanglement entropy of $\rho_A := \mathrm{Tr}_{A^c} |\Psi_0\rangle\langle\Psi_0|$:*

$$S(\rho_A) \le (c_d \ln D)\,(1+r)^{d-1} \cdot \ell_0, \tag{4}$$

*where $c_d$ is a constant depending only on the dimension $d$ of the lattice, $D$ is the maximum dimension of the on-site Hilbert spaces and $\ell_0 = \min\{\ell : \Delta_0(\ell) \le \ell/(1+r)\}$.*

# Local Gaps.

## Definition

**Local-Gap:** We define $\mathbf{H_0}$ to be **locally gapped** w.r.t. a function $\gamma(r)$, if $\mathbf{H_B} \geq \gamma(\mathbf{r})(\mathbf{1} - \mathbf{P_B})$, where $B = b_u(r)$. If $\gamma(r)$ decays at most **polynomially**, we say that $H_0$ satisfies the **Local-Gap** condition.

**Open Problem:** Is the **Local-Gap** condition always satisfied if $H_0$ is a sum of local projections with frustration-free ground-state and a spectral gap?

**Open Problem:** Is **Local-TQO** important for **Local-Gap** in this setting?

# A brief History of Stability...

1. (Euclid, 314 B.C., Kato, '66) Let $H_0$ have spectral gap $\gamma > 0$ and **unique** groundstate. Then, $H_0 + V$ retains a gap if $\|V\| < \gamma/2$.

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3. (Bravyi, Hastings, M., '10) $H_0$ is sum of **commuting projections**, with spectral gap $\gamma$ and **frustration-free** groundstate subspace, satisfying a form of **Local Topological Order**. Then, for $V$ a sum of **rapidly decaying terms** $V_u$, there exists a $J_0$ such that for $\|V_u\| \leq J_0 \implies$ **stable gap**. (common eigenbasis)

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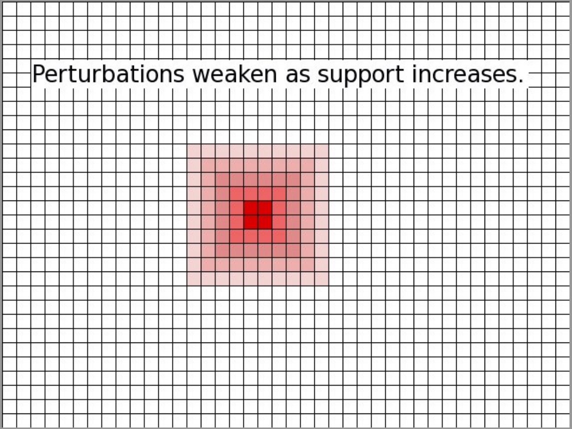
4. (M., Pytel, '11) Let $H_0$ have gap $\gamma$ and **frustration-free** groundstate subspace, satisfying **Local-TQO** and **Local-Gap**. Then, stability holds for all perturbations $V$, as above. (common groundstate)

# Decaying perturbations...




For each site $u \in \Lambda$, we allow perturbations supported on $b_u(r)$. As the radius of the support increases, the norm of the perturbation decreases rapidly.

# The Perturbations: Local decomposition and strength.

Definition

We say that $V$ **has strength** $J$ **and rapid decay** $f$, if we can write

$$V = \sum_{u \in \Lambda} V_u, \quad V_u := \sum_{r \geq 0} V_u(r),$$

such that $V_u(r)$ has support on $b_u(r)$ and $\|V_u(r)\| \leq J\, f(r)$, $r \geq 0$.

# The main result.

Theorem

- *Let $H_0$ be a* **frustration-free** *Hamiltonian satisfying* **Local-TQO** *and* **Local-Gap** *with decay given by $\Delta_0(r)$ and $\gamma(r)$, respectively.*

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- *Let $V$ be a* **strength** *$J$ perturbation, with* **decay** *$f(r)$.*
- *Then, $H_0 + V$ has* **spectral gap bounded below** *by*

$$(1 - c_0 J)\gamma - c_1 \, J \, L^d \left( \sqrt{\Delta_0(L^*)} + w(L^*) \right),$$

*where $c_0 = \sum_{r=1}^{L} r^d \cdot [w(r)/\gamma(r)]$ and $w(r)$ has rapid decay related to the decay rate of $f(r)$. The* **groundstate splitting** *is bounded above by $J \, L^d \left( \sqrt{\Delta_0(L^*)} + w(L^*) \right)$. Since $L^* \sim L^\alpha$, this implies exponentially small splitting for rapidly decaying $\Delta_0$ and $w$.*

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- *For* $f(r)$ **decaying exponentially**, $w(r)$ *decays* **sub-exponentially**.

# Spectral Flow - Hastings' Quasi-Adiabatic Evolution

Definition

Define the unitary operator $U_s$ (**Spectral Flow**), by:

$$\partial_s U_s \equiv i\mathcal{D}_s\, U_s, \quad U_0 = \mathbb{1}, \tag{5}$$

where $\mathcal{D}_s$ simulates the **generator of the adiabatic evolution** for a family of gapped Hamiltonians $H_s$. (Next slide.)

# Generators of quasi-adiabatic evolution.

Definition

For $H_s = H_0 + sV$, define the quasi-adiabatic evolution generator $\mathcal{D}_s$ by:

$$\mathcal{D}_s \equiv \int_{-\infty}^{\infty} s_\gamma(t) \left( \int_0^t e^{iuH_s}(V)e^{-iuH_s} \mathrm{d}u \right) \mathrm{d}t, \tag{6}$$

where the function $s_\gamma(t)$ (called a **filter function**) is chosen to satisfy the following properties:

1. The Fourier transform of $s_\gamma(t)$, denoted by $\hat{s}_\gamma(\omega)$, obeys:

$$|\omega| \geq \gamma/2 \quad \rightarrow \quad \hat{s}_\gamma(\omega) = 0 \quad \text{(compact support)}. \tag{7}$$

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2. $s_\gamma(t)$ decays like $\exp\{-\frac{\gamma|t|}{4\log^2 \gamma|t|}\}$ (sub-exponential decay).

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where the function $s_\gamma(t)$ (called a **filter function**) is chosen to satisfy the following properties:

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4. Note: This magical function $s_\gamma(t)$ exists and can be quite the ice-breaker on a first date.

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- *The spectral flow satisfies* $\|\mathbf{U_s - U_A \otimes U_{\Lambda\setminus A} U_{\partial A(\ell)}}\| \leq \mathbf{\Delta}(\ell)$, *where $\Delta$ decays sub-exponentially.*

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- Using the **spectral flow**, unitarily transform the gapped family of Hamiltonians $H_s := H_0 + sV$ into $U^\dagger(s)H_sU(s) = H_0 + V'$, so that $[V', P_0] = 0$. (global block-diagonality)

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- **Relative boundedness** implies that $H_0 + W + \Delta$, has a spectral gap, which is equivalent to the stability of the spectrum of $H_0 + V$. (unitary invariance + global energy shift)

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- Is there a notion of **frustration-free parent Hamiltonian**? Do they have **optimal Local-TQO decay** for given $\mathbf{P_0}$?

# Questions?


Figure: Charlie on xkcd.

# Thank you, Montreal!