Topological quantum codes & Fault-tolerance

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Outline

- Motivation

- Topological codes, Example 1: Surface codes

- Remarks

Fault-tolerant quantum computation

- *Fault-tolerance is the art of maintaining the quantum speedup in the presence of decoherence.*

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*Fault-tolerance theorem**: If for a universal quantum computer the noise per elementary operation is below a constant non-zero <u>error threshold</u> ϵ then arbitrarily long quantum computations can be performed <u>efficiently</u> with arbitrary accuracy.

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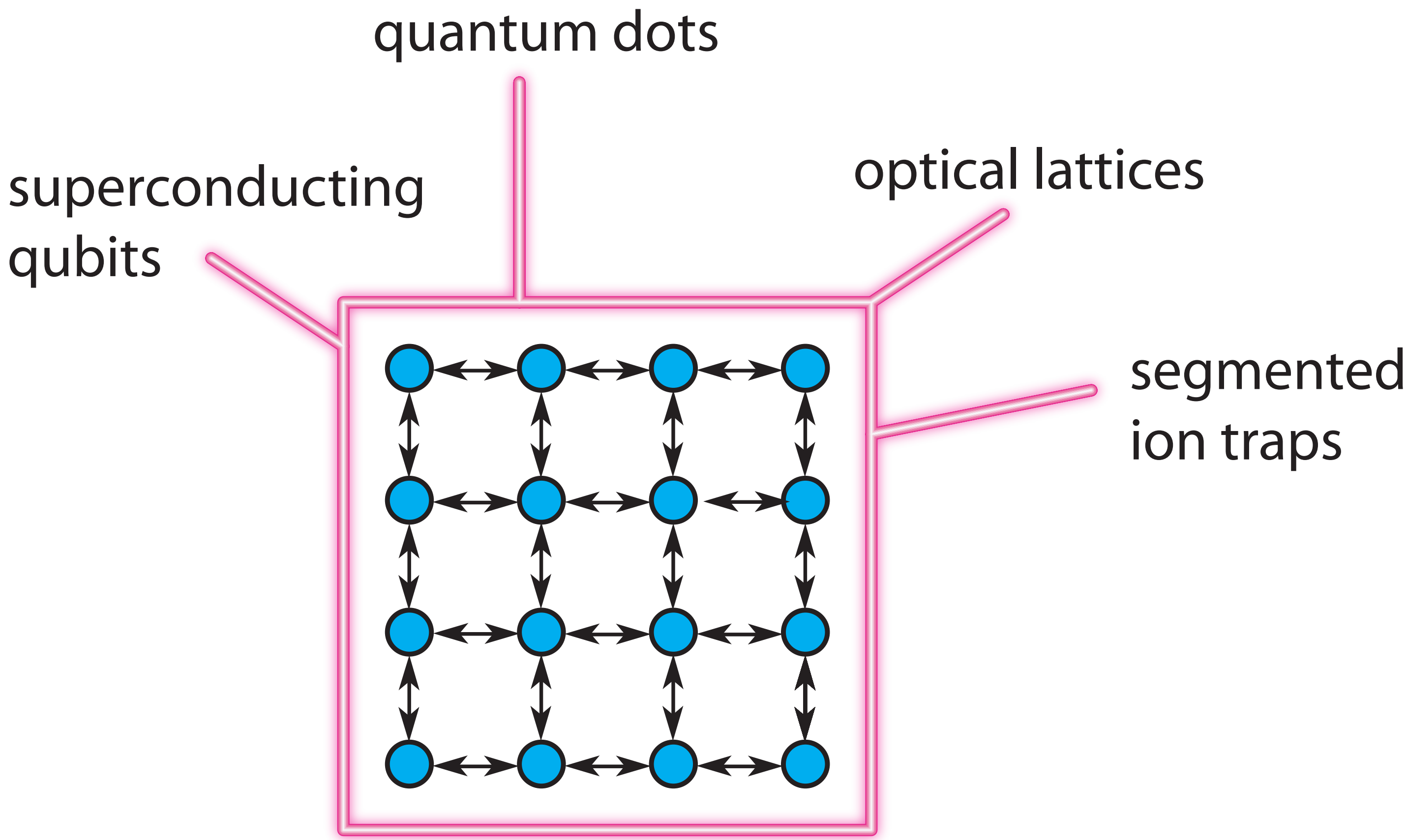
*: Aharonov & Ben-Or (1996), Kitaev (1997), Knill & Laflamme & Zurek (1998), Aliferis & Gottesman & Preskill (2005)

Our setting



- 2D, *nearest-neighbor translation-invariant interaction*.
- High fault-tolerance threshold
- Moderate overhead scaling

Known threshold values

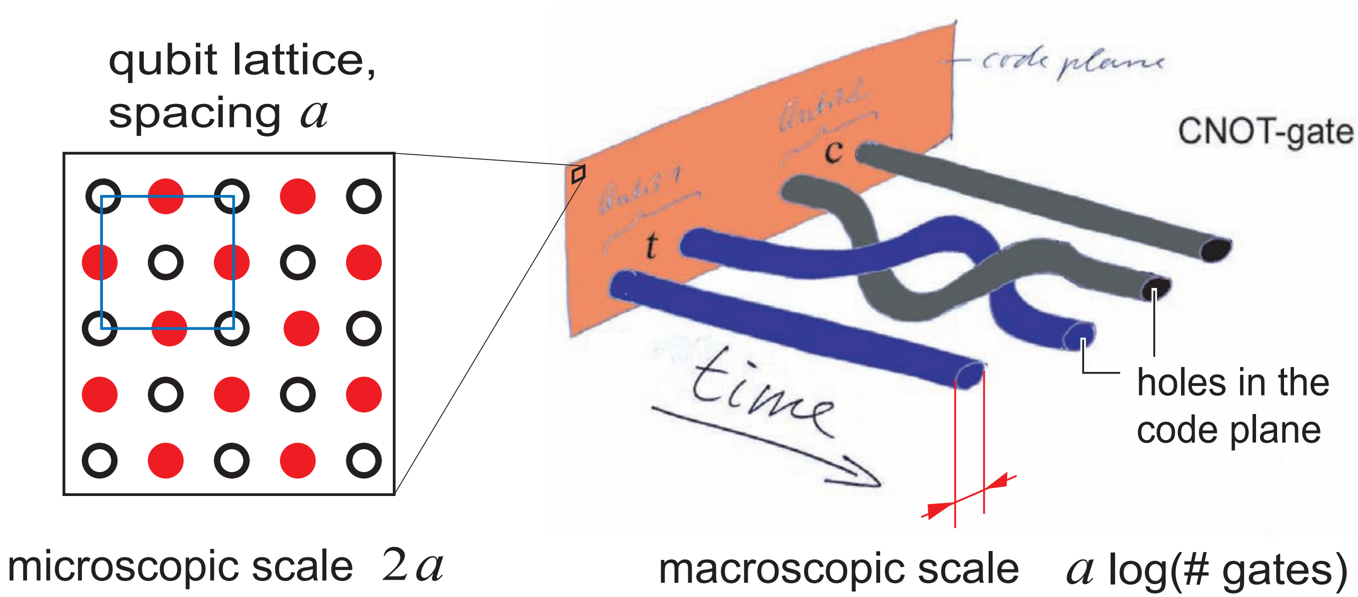
no constraint	geometric constraint: 2D	geometric constraint: 1D
[1] — 0.03, est.	[5] — $7 \cdot 10^{-3}$, est.	
[2] — 10^{-3}, est.	[6] — $2 \cdot 10^{-5}$, bd.	
[3] — 10^{-4}, est.		[7] — 10^{-8}, bd.
[4] — 10^{-5}, bd.		

- Error sources:
  $|+\rangle$-Preparation, $\Lambda(Z)$-gates, Hadamard gates, measurement.

[1] Knill, (2005); [2] Zalka (1999); [3] Dawson & Nielsen (2005); [4] Aliferis & Gottesman & Preskill (2005), [5] Raussendorf & Harrington, quant-ph/0610062; [6] Svore & DiVincenzo & Terhal, quant-ph/0604090, [7] Aharonov & Ben-Or (1999)

Main ingredients

1. Fault-tolerance from surface codes [Kitaev 97]: Translation-invariant and short-range interaction.
2. *Topological quantum gates via time-dependent boundary conditions.*

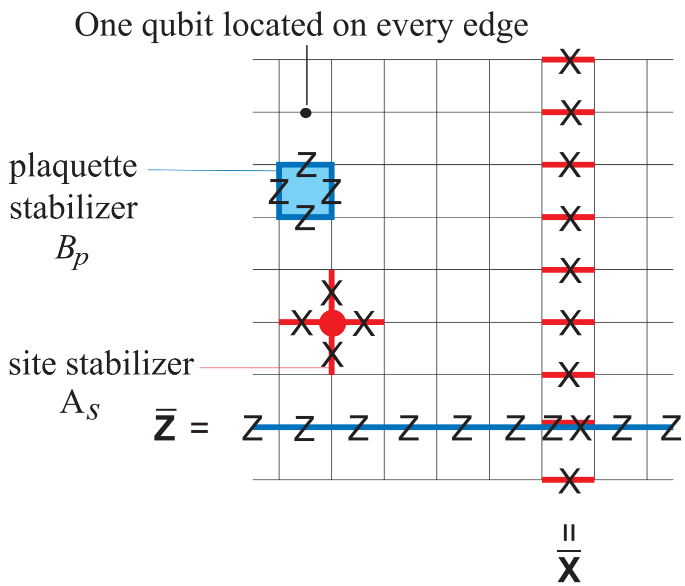


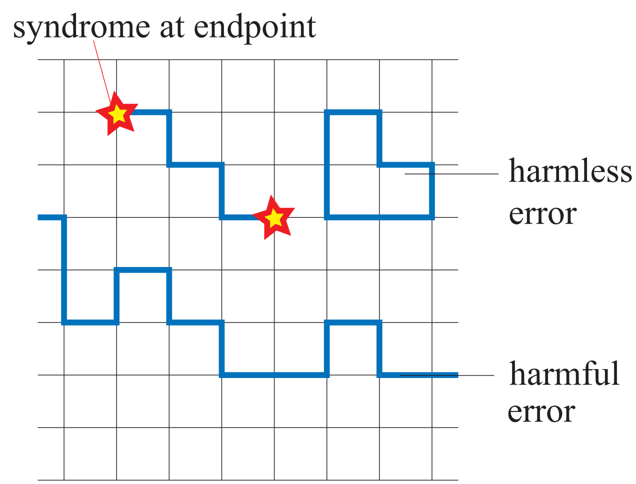
microscopic view   macroscopic view

Part I:

Microscopic view: Fault-tolerance

1.1 The surface code



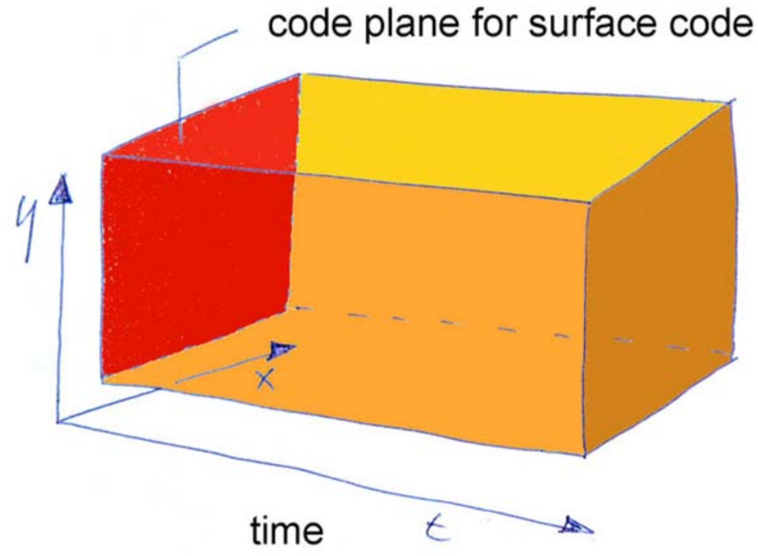


$$|\psi\rangle = A_s|\psi\rangle = B_p|\psi\rangle, \ \forall|\psi\rangle \in \mathcal{H}_C, \forall s, p. \tag{1}$$

- Surface codes are stabilizer codes associated with 2D lattices.
- Only the *homology class* of an error chain matters.

A. Kitaev,quant-ph/9707021 (1997).

1.2 Topological error correction

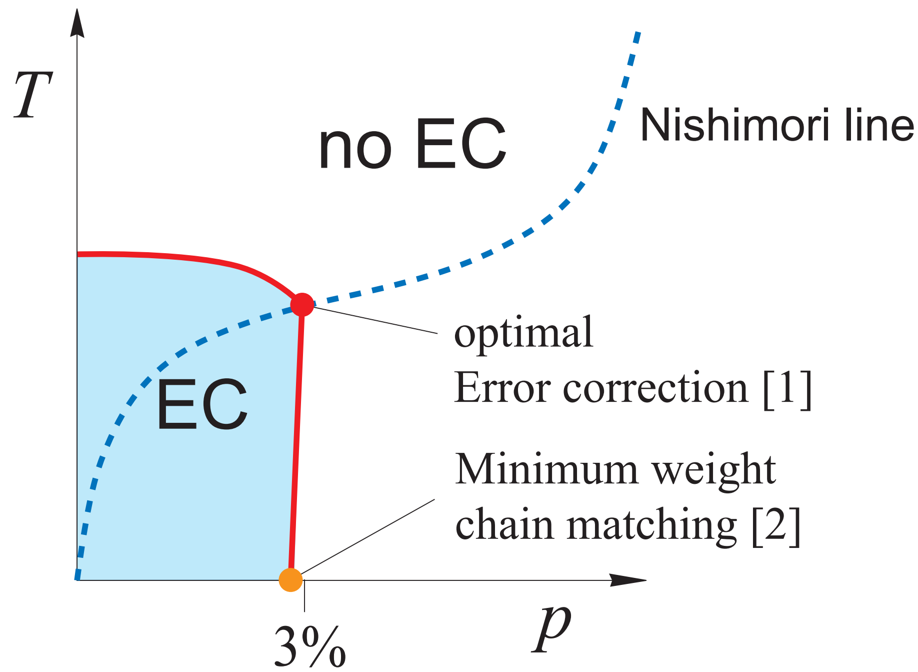


- Fault-tolerant data storage with planar code described by *Random plaquette* Z_2*-gauge model* (RPGM) [1].

[1] Dennis et *al.*, quant-ph/0110143 (2001).

1.3 Phase diagram of the RPGM

Map error correction to statistical mechanics:



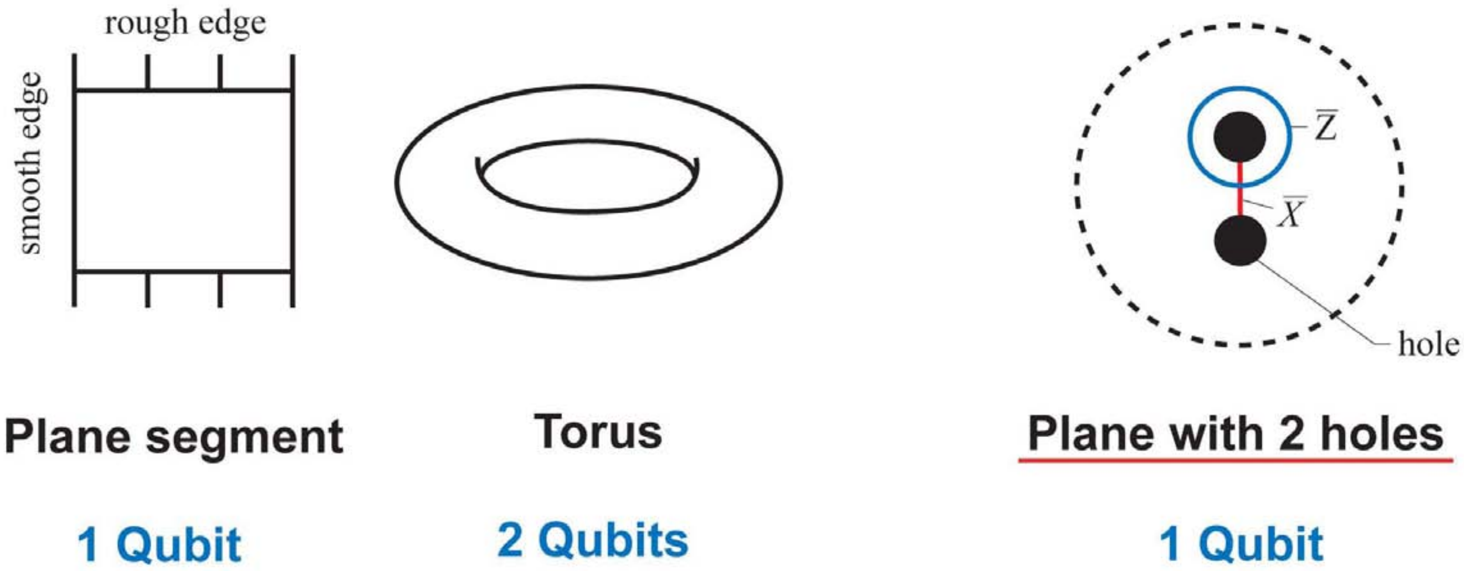
- Have an error budget of 3%.

[1] T. Ohno et *al.*, quant-ph/0401101 (2004). [2] E. Dennis et *al.*, quant-ph/0110143 (2001); J. Edmonds, Canadian J. Math. **17**, 449 (1965).

Part II:

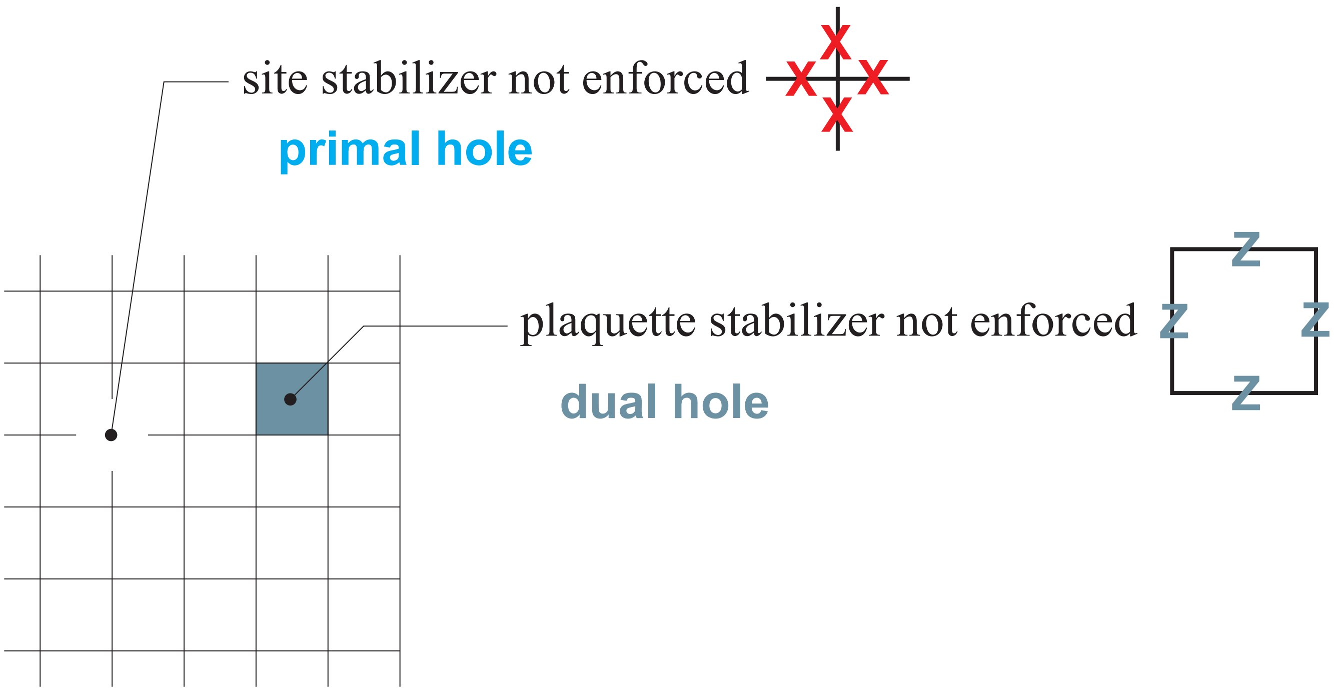
Macroscopic view: Protected quantum gates

2.1 How to encode qubits



- Storage capacity of the code depends upon the topology of the code surface.

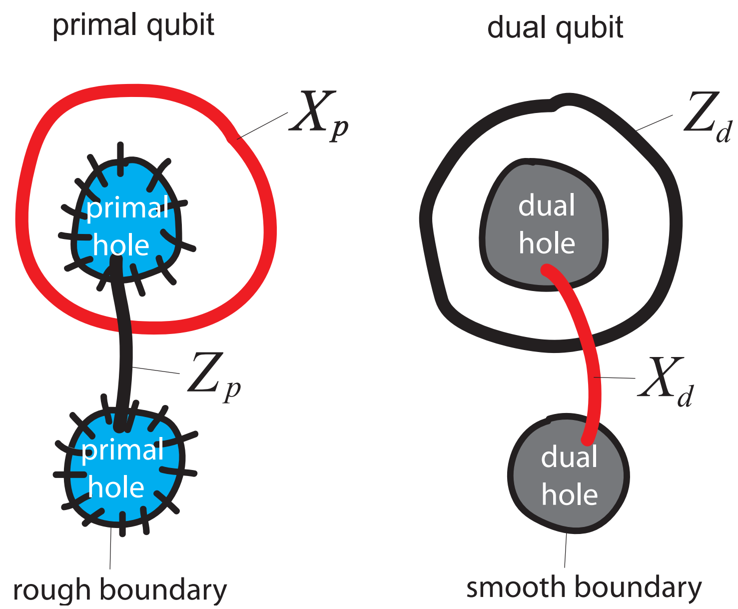
2.2 Surface code on plane with holes



- There are two types of holes: primal and dual.
- A pair of same-type holes constitutes a qubit.

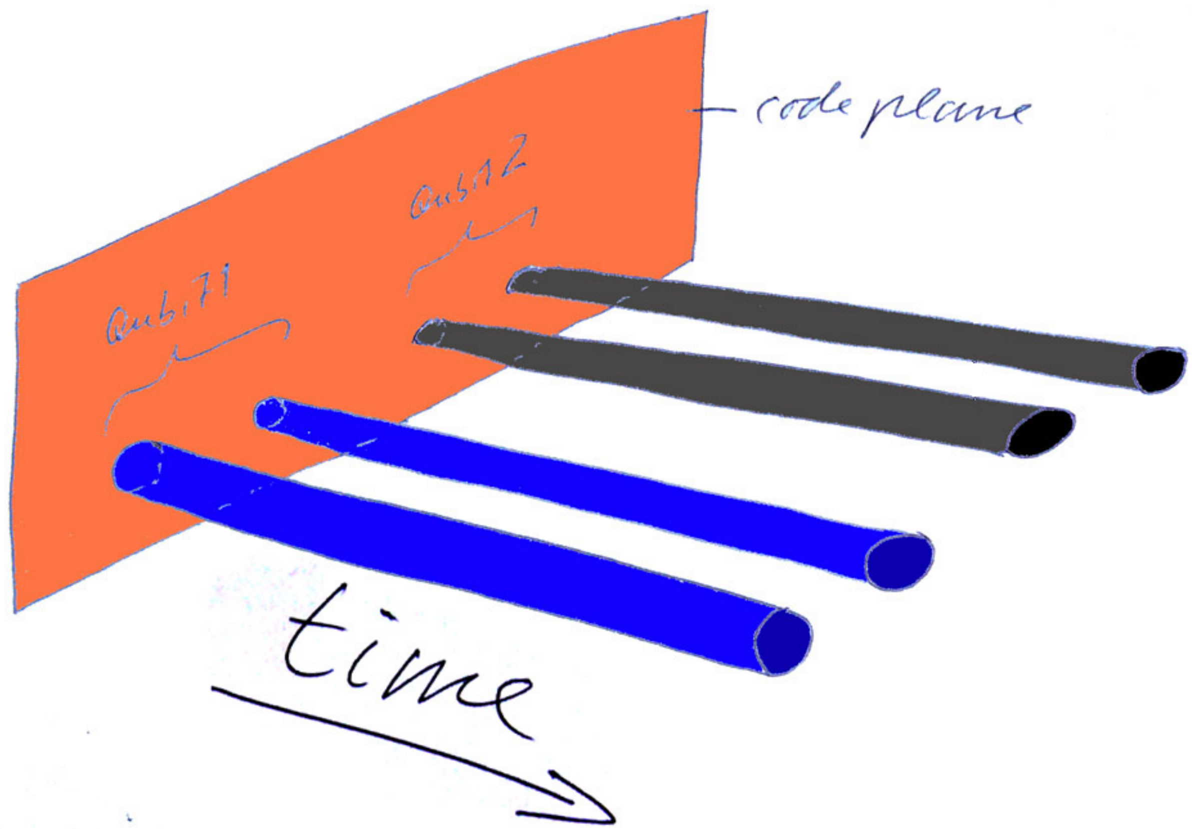
2.2 Surface code on plane with holes

Surface code with boundary:



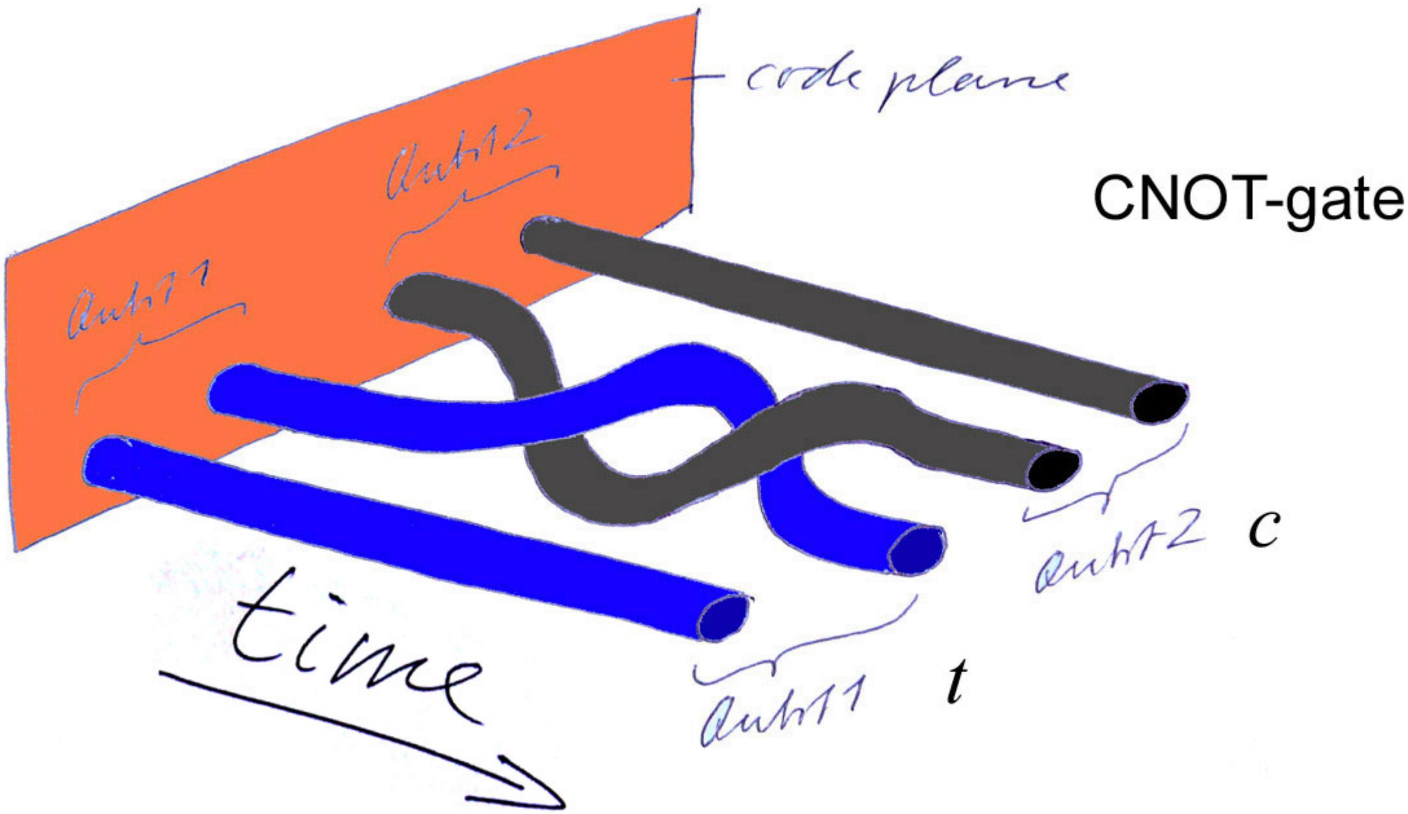
- X-chain cannot end in primal hole, can end in dual hole.
- Z-chain can end in primal hole, cannot end in dual hole.

2.3 Encoded quantum gates



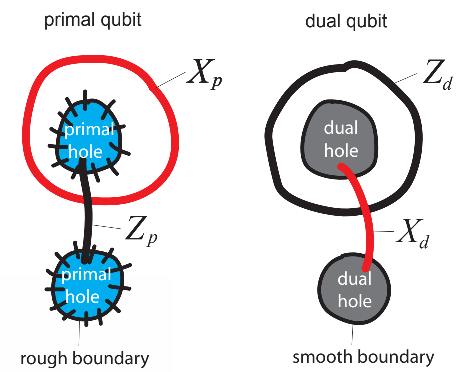
Now consider worldlines of holes.

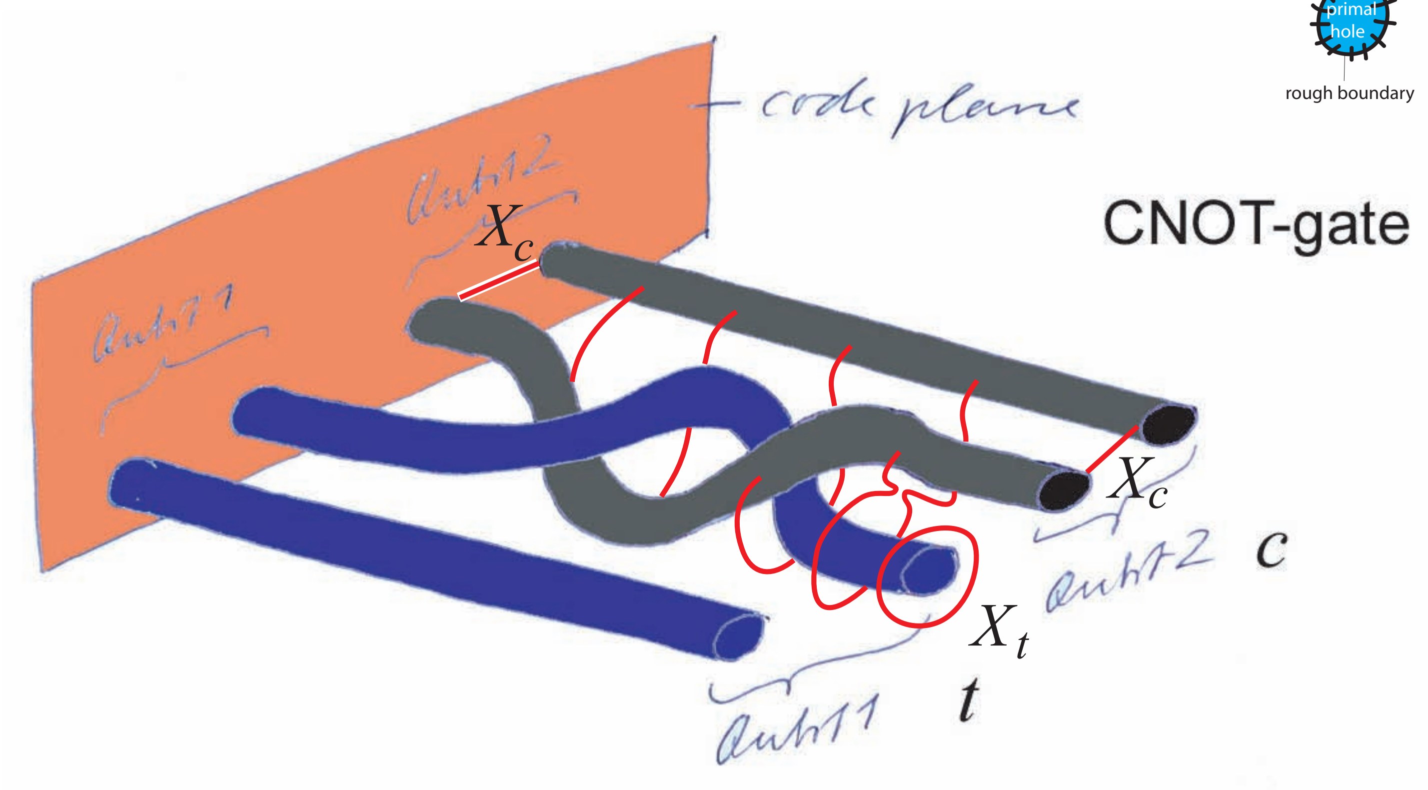
2.3 Encoded quantum gates



Topological quantum gates are encoded in the way worldlines of primal and dual holes are braided.

2.4 A CNOT-gate



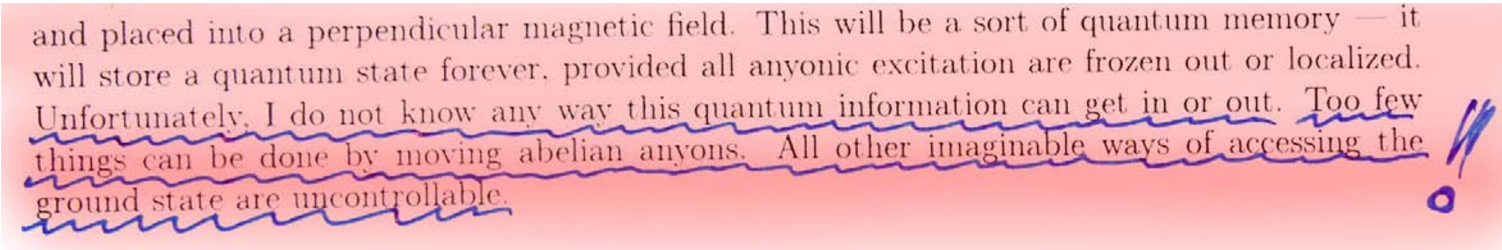


- Propagation relation: $X_c \longrightarrow X_c \otimes X_t$.
- Remaining prop rel $Z_c \rightarrow Z_c$, $X_t \rightarrow X_t$, $Z_t \rightarrow Z_c \otimes Z_t$ for CNOT derived analogously.

2.5 Non-abelian gates with surface codes

Problem:

Braiding-like operations in surface codes only yield *abelian* gates.


and placed into a perpendicular magnetic field. This will be a sort of quantum memory — it will store a quantum state forever, provided all anyonic excitation are frozen out or localized. Unfortunately, I do not know any way this quantum information can get in or out. Too few things can be done by moving abelian anyons. All other imaginable ways of accessing the ground state are uncontrollable.

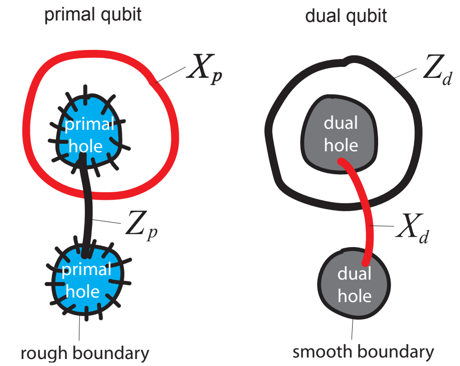
A. Kitaev, quant-ph/970721

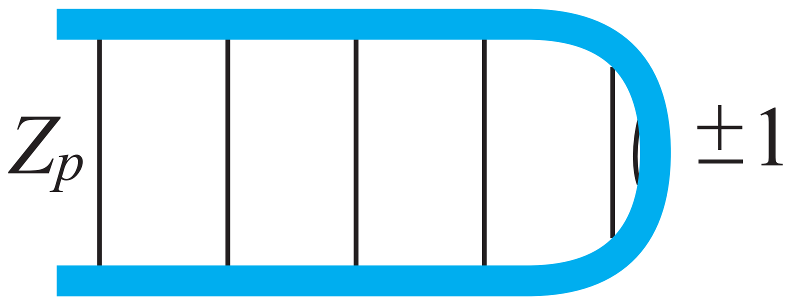
Solution:

Non-abelian gates are realized by *changing the topology* of the code surface with time.

Another example of this: M. Freedman, Ch. Nayak, K. Walker, cond-mat/0512072.

2.5 Z-measurement on primal qubit





2.5 Non-abelian gates with surface codes

- Starting point: CNOT with dual control, primal target,

Surface topology unchanged - commuting gates.

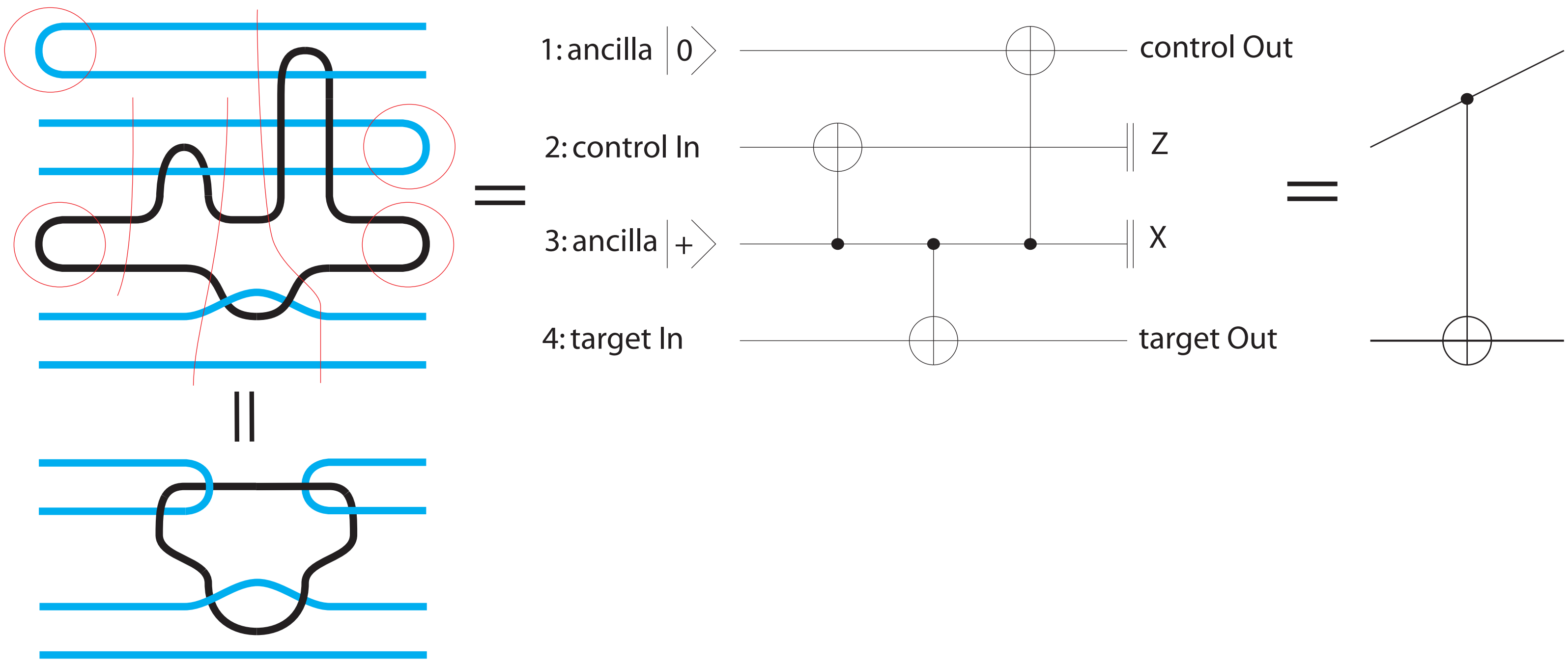
- Add on:

surface topology changed - non-commuting gates.

- *Can we compose a general CNOTs out of these operations?*

2.5 Non-abelian gates with surface codes

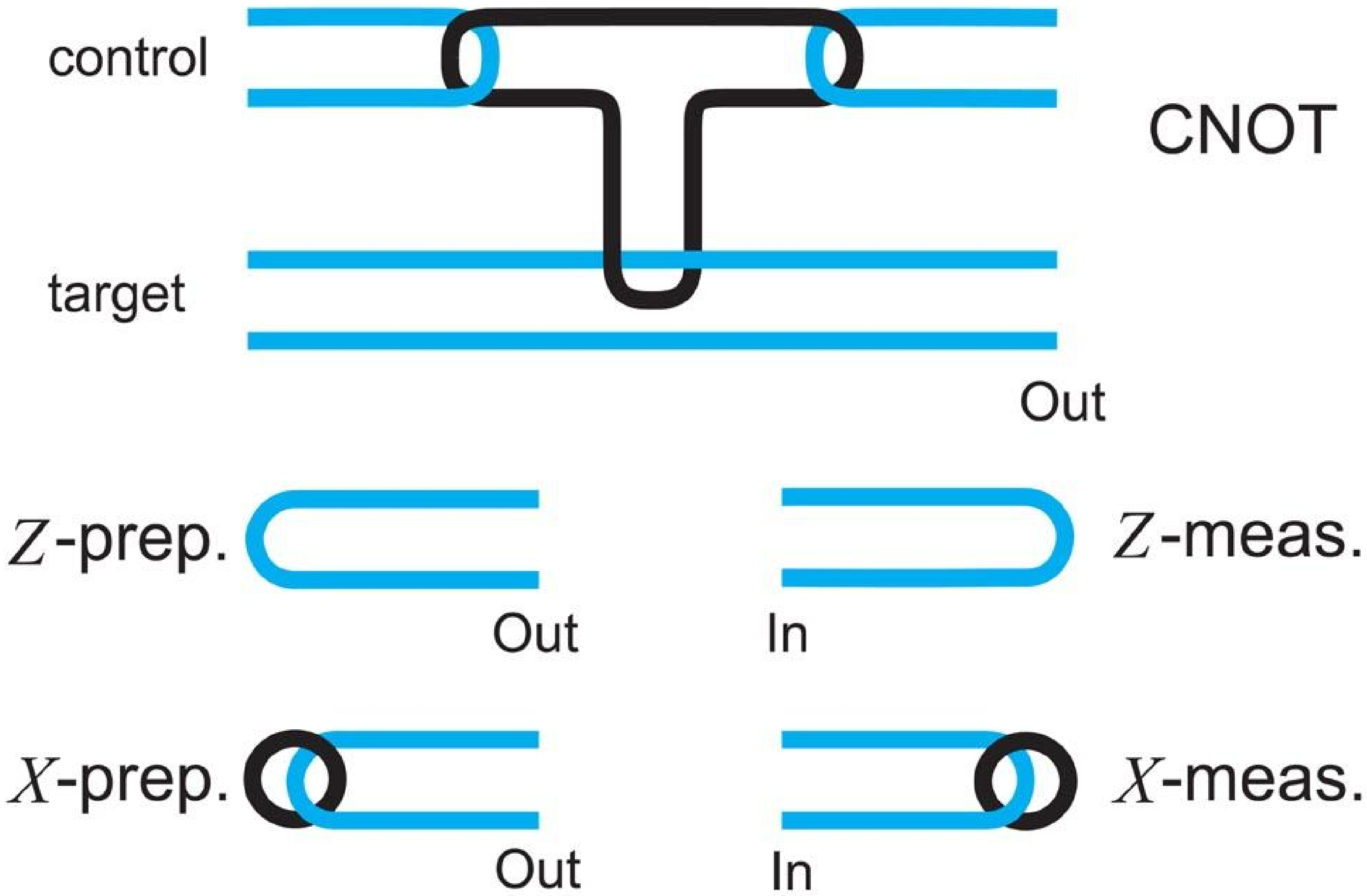
- **Yes.**



CNOT-gate between two primal qubits in either direction.

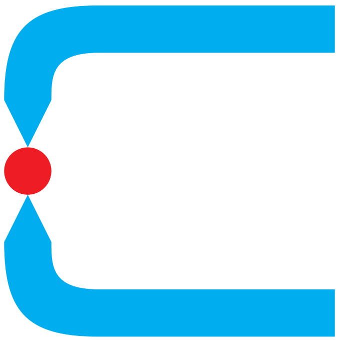
$\longrightarrow$ **nonabelian set of gates.**

2.6 Topological quantum gates



2.7 Universal gate set

- Need one non-Clifford element:
  fault-tolerant preparation of $|A\rangle := \frac{X+Y}{\sqrt{2}}|A\rangle$.



Singular Qubit

- FT prep. of $|A\rangle$ provided through realization of *magic state distillation**.

*: S. Bravyi and A. Kitaev, Phys. Rev. A 71, 022316 (2005).

Part III:

Threshold and Overhead Scaling

3.1 Error Model

Error sources:

1. $|+\rangle$-preparation: Perfect preparation followed by 1-qubit partially depolarizing noise with probability p_P.
2. $\Lambda(Z)$-gates (space-like edges of $\mathcal{L}$): Perfect gates followed by 2-qubit partially depolarizing noise with probability p_2.
3. Hadamard-gates (time-like edges of $\mathcal{L}$): Perfect gates followed by 1-qubit partially depolarizing noise with probability p_1.
4. Measurement: Perfect measurement preceeded by 1-qubit partially depolarizing noise with probability p_M .

- No qubit is ever idle. (Additional memory error - same threshold)
- For threshold set $p_1 = p_2 = p_P = p_M =: p$.

3.2 Fault-tolerance threshold

Topological threshold in cluster region V:

$$p_c = 7.5 \times 10^{-3}. \quad (2)$$

Purification threshold for fault-tolerant $|A\rangle$-preparation:

$$p_c = 3.7 \times 10^{-2}. \quad (3)$$

Topological EC sets the overall threshold.

3.3 Overhead and Robustness

- Denote by S (S') the bare (encoded) size of a quantum circuit. Then, for the described method:

$$S' \sim S \log^3 S. \quad (4)$$

- The threshold is robust against variations in the error model such as higher weight elemetary errors, long-distance errors.

Summary

Scenario:

- Local and next-neighbor gates in 2D single layer structure.

Numbers:

- Fault-tolerance threshold of 7.5×10^{-3} for preparation, gate, memory and measurement error (each source).

Suitable systems for realization:

- Cold atoms in optical lattices, segmented ion traps, Josephson junction arrays, ...

Reading

- Topological quantum codes:

  Surface codes: A. Kitaev, Ann. Phys. (NY) 303,2 (2003).
  Color codes: H. Bombin and M.A. Martin-Delgado, PRL 97, 180501 (2006).
  Color subsystem codes: H. Bombin, PRA 81, 032301 (2010).

- Error-correction algorithms and Stat. Mech. connection:

  E. Dennis, A. Kitaev, A. Landahl, J. Preskill, quant-ph/0110143 (2001).
  G. Duclos-Cianci, D. Poulin, PRL 104,050504 (2010).
  H. Bombin, G. Duclos-Cianci, D. Poulin, arXiv:1103.4606v1 (2011).

- Quantum computation with topological codes:

  R. Raussendorf and J. Harrington, PRL 97, (2007).
  H. Bombin, PRL 105, 030403 (2010).
  H. Bombin, New J. Phys. 13, 043005 (2011).