Quantum Information and Quantum Many-body Systems

– Lecture 1 –

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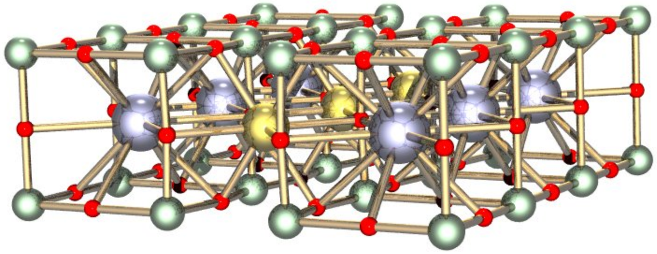
Quantum Information and Quantum Many-Body Systems

- Aim: Understand the physics of **quantum systems** composed of **many particles** (systems)

- In many cases, quantum correlations between particles are not relevant (mean field theory)

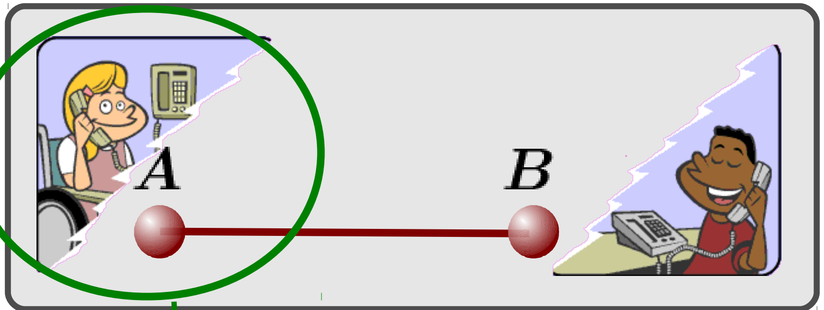
- **Strong correlations** involved
  $\Rightarrow$ **entanglement** becomes important



- Entanglement Theory:
  - central part of quantum information theory
  - how can we **measure entanglement**?
  - **what can we do** with entanglement,
    and **what is impossible**?

Can we use quantum information techniques (in particular entanglement theory) to obtain a better understanding of quantum many-body systems?

Entanglement



- two (and more) qubits: **entanglement**

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\Big[|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\Big]$$

- **How much entanglement** is in some state

  e.g. $|\phi\rangle = \alpha|0\rangle_A|0\rangle_B + \beta|1\rangle_A|1\rangle_B$?

  $\leftrightarrow$ How much **perfect entanglement** $|\Psi^+\rangle$ does it contain?

- **reduced state** of Alice $\rho_A := \mathrm{tr}_B|\phi\rangle\langle\phi|$:

$$\rho_A = |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$$

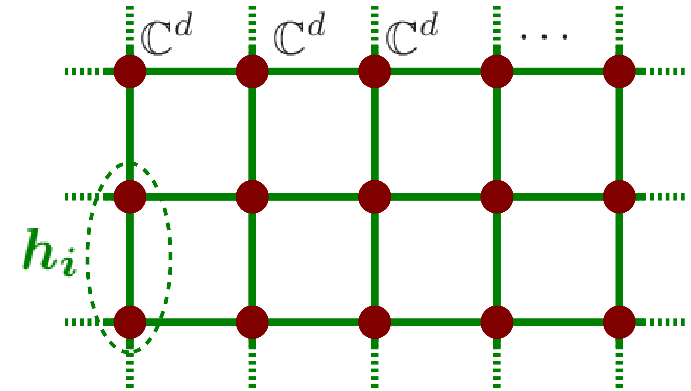
- more entanglement $\leftrightarrow$ **more uncertainty** in ρ_A
- measure of uncertainty (entanglement): **von Neumann entropy**

$$S(\rho_A) = -\mathrm{tr}[\rho_A \log \rho_A]$$

  $\Rightarrow$ provides **quantitative measure** of entanglement

entropy = entanglement

Quantum many-body systems



- We consider systems composed of many (N)

  d-level systems $|0\rangle, |1\rangle, \ldots, |d-1\rangle$ ("spins")

  with a **locality notion** ($\to$ lattice geometry)

- Behavior of system described by **local Hamiltonian**

$$H = \sum_{i=1}^{M} h_i$$

acts on small region only (e.g. nearest neighbors)

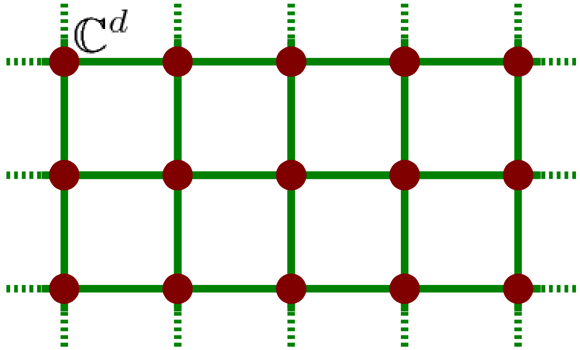
with $h_i = h_i^\dagger$ a local operator: $h_i \equiv \mathbb{1} \otimes h_i \otimes \mathbb{1}$

- Our focus: **ground state $|\Psi_0\rangle$:**

$$H|\Psi_0\rangle = E_0|\Psi_0\rangle \quad \text{with } E_0 \text{ **smallest eigenvalue** } \lambda_{\min}(H) \text{ of } H$$

- **Questions:** - What is E_0 ?
  - Which properties does $|\Psi_0\rangle$ have?
    (such as correlation functions $\langle\Psi_0|\mathbb{1} \otimes A_i \otimes \mathbb{1} \otimes B_j \otimes \mathbb{1}|\Psi_0\rangle$)
  - How do these things depend on H ?

How hard is it to describe the ground state?

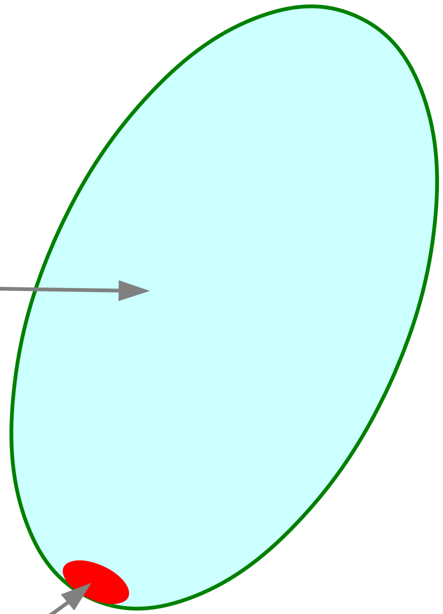


- N spins, $H = \sum_{i=1}^{M} h_i$
- can we **describe the ground state** $|\Psi_0\rangle$?

- Problem for large N:

$$|\Psi_0\rangle = \sum_{i_1,\ldots,i_N} c_{i_1\cdots i_N} |i_1,\ldots,i_N\rangle \;\in\; (\mathbb{C}^d)^{\otimes N} = \mathbb{C}^{(d^N)}$$

exponentially large
Hilbert space $\mathbb{C}^{(d^N)}$!



- But there is hope:

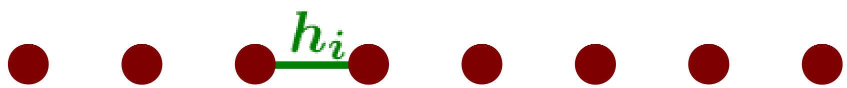
$\boldsymbol{H = \sum_{i=1}^{M} h_i}$ has only $\boldsymbol{M \propto N}$ **parameters**

$\rightarrow |\Psi_0\rangle$ lives in **small region** of Hilbert space

Can we find an **efficient description of ground states**
from which we can **efficiently compute quantities of interest**?

Physical Hamiltonians

- **Physical guideline** for suitable **ansatz states**?
- Focus (for the moment) on **one-dimensional (1D) systems**



with **local Hamiltonian** $H_N = \sum_{i=1}^{N} h_i$

- H_N **uniform family** of Hamiltonians:

acts beween spin i and $i+1$

  - E.g., **translational invariant** $h_i = \mathbb{1} \otimes h \otimes \mathbb{1}$

    [either periodic boundary conditions (PBC) or open boundary conditions (OBC)]

- **spectral gap** of H :

second smallest eigenvalue

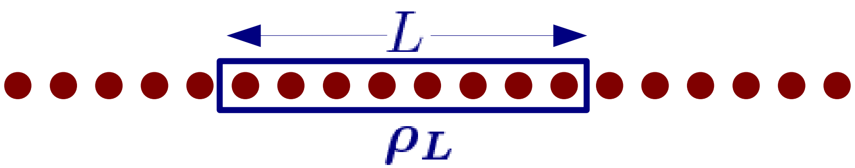
smallest eigenvalue

$$\Delta(H) := \lambda_2(H) - \lambda_1(H)$$

- **gapped Hamiltonians:** $\Delta(H_N) \geq \Delta_0 > 0$ uniform in N

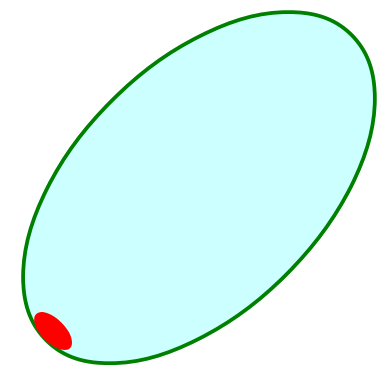
The area law

- What can we say about ground states of gapped Hamiltonians?
- **Area law** for ground states of **gapped Hamiltonians**:



$$S(\rho_L) \sim \mathrm{const.}$$

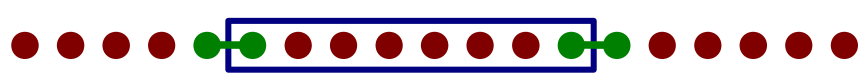
entropy $S(\rho_L)$ is bounded by a **constant**



- Suprising: for random states, we expect $S(\rho_L) \sim L$
- Even for **gapless systems:** $S(\rho_L) \sim \log L$

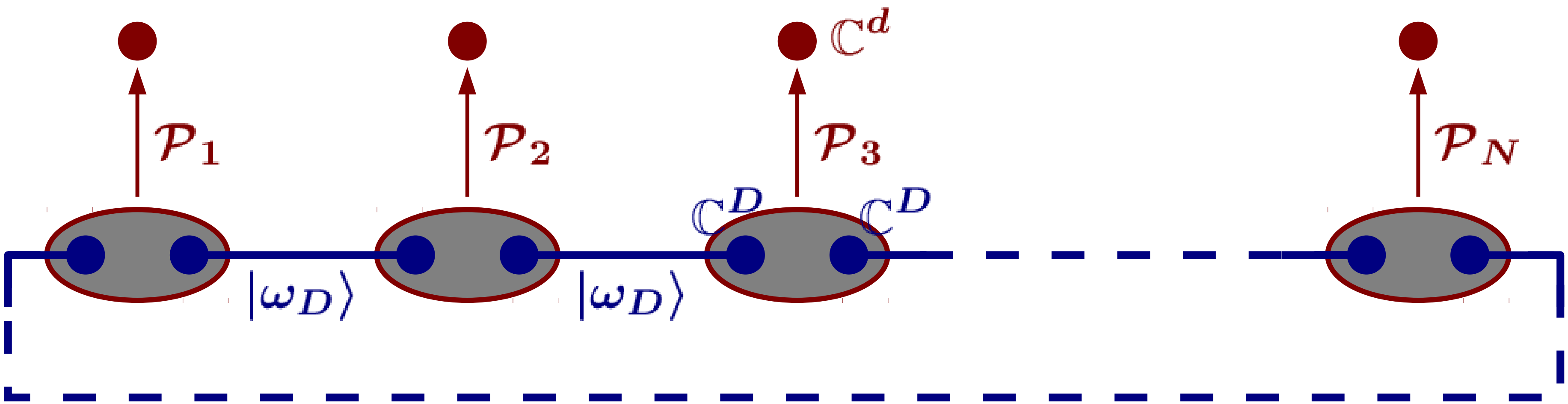
- Quantum Information: entropy ≡ entanglement

$\Rightarrow$ **entanglement** located around the **boundary**



$\Rightarrow$ construct ansatz from **entanglement** between **adjacent sites**

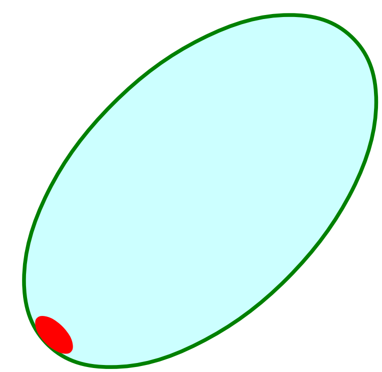
An ansatz for states with an area law



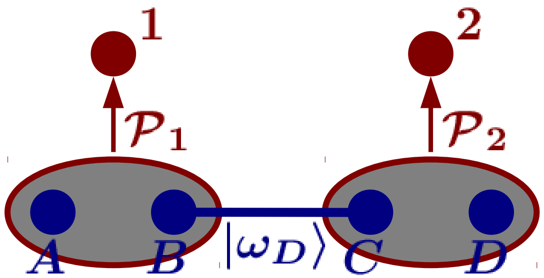
- each site composed of two **auxiliary particles** (“virtual particles”) forming max. entangled **bonds** $|\omega_D\rangle := \sum_{i=1}^{D} |i,i\rangle$ (D: “bond dimension”)
- apply **linear map** (“projector”) $\mathcal{P}_k : \mathbb{C}^D \times \mathbb{C}^D \to \mathbb{C}^d$

$$\Rightarrow \quad |\psi\rangle = (\mathcal{P}_1 \otimes \cdots \otimes \mathcal{P}_N)|\omega_D\rangle^{\otimes N}$$

- satisfies **area law** by construction
- state characterized by $\mathcal{P}_1, \ldots, P_N \rightarrow NdD^2$ parameters
- family of states: enlarged by increasing D



Formulation in terms of Matrix Products



$$\mathcal{P}_s = \sum_{i,\alpha,\beta} A^{[s],i}_{\alpha\beta} |i\rangle\langle\alpha,\beta|$$

$A^{[s],i}$: $D \times D$ matrices

$$(\mathcal{P}_1 \otimes \mathcal{P}_2)|\omega_D\rangle = \Big[\sum_{i,\alpha,\beta} A^{[1],i}_{\alpha\beta}|i\rangle_1\langle\alpha,\beta|_{AB}\Big]\Big[\sum_{j,\gamma,\delta} A^{[2],j}_{\gamma\delta}|j\rangle_2\langle\gamma,\delta|_{CD}\Big]\Big[\sum_k |k,k\rangle_{BC}\Big]$$

$$= \sum_{i,j,\alpha,\delta}\Big[\sum_\beta A^{[1],i}_{\alpha\beta}A^{[2],j}_{\beta\delta}\Big]|i,j\rangle_{12}\langle\alpha,\delta|_{AD} \qquad \boldsymbol{\beta = \gamma}$$

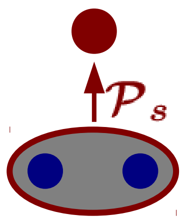
$$= \sum_{i,j,\alpha,\delta} \boldsymbol{(A^{[1],i}A^{2,j})_{\alpha\delta}}|i,j\rangle_{12}\langle\alpha,\delta|_{AD}$$

- iterate this for the whole state $|\psi\rangle = (\mathcal{P}_1 \otimes \cdots \otimes \mathcal{P}_N)|\omega_D\rangle^{\otimes N}$:

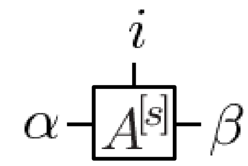
$$|\psi\rangle = \sum_{i_1,\dots,i_N} \mathrm{tr}[A^{[1],i_1}A^{[2],i_2}\dots A^{N,i_N}]|i_1,\dots,i_N\rangle$$ **“Matrix Product State”**

(or $|\psi\rangle = \sum_{i_1,\dots,i_N} \langle l|A^{[1],i_1}A^{[2],i_2}\cdots A^{[N],i_N}|r\rangle|i_1,\dots,i_N\rangle$ for open boundaries)

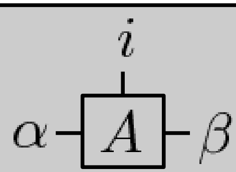
Formulation in terms of Tensor Networks

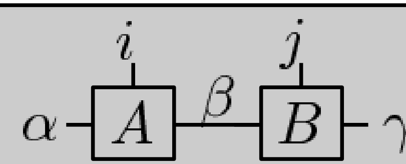


$$\mathcal{P}_s = \sum_{i,\alpha,\beta} A^{[s],i}_{\alpha,\beta} |i\rangle\langle\alpha,\beta|$$

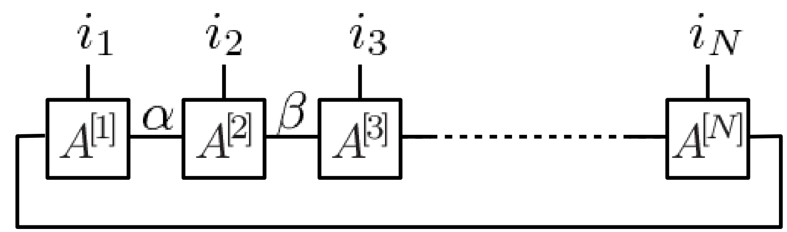
$A^{[s],i}_{\alpha\beta} \equiv$ 

- **Tensor Network** notation:

$A^i_{\alpha\beta} \equiv$ 

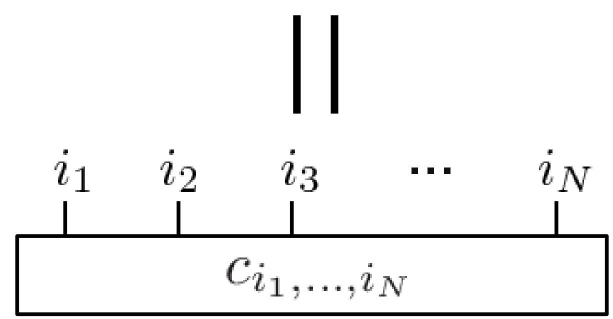
$\sum_\beta A^i_{\alpha\beta} B^j_{\beta\gamma} \equiv$ 

$$\mathbf{tr}[A^{[1],i_1} A^{[2],i_2} \cdots A^{[N],i_N}] =$$



- Matrix Product States can be written as

$$|\psi\rangle = \sum_{i_1,\ldots,i_N} c_{i_1,\ldots,i_N} |i_1,\ldots,i_N\rangle$$ with



"Tensor Network States"

Examples

• any **product state**, e.g.: $(\alpha|0\rangle + \beta|1\rangle)^{\otimes N}$

$\underline{D=1:}$ $\quad \boldsymbol{A^0 = (\alpha)\ ;\ A^1 = (\beta)}$

$$|\psi\rangle = \sum \mathrm{tr}[A^{i_1} A^{i_2} \dots A^{i_N}] |i_1, \dots, i_N\rangle \qquad \boldsymbol{\mathcal{P} = (\alpha|0\rangle + \beta|1\rangle)}$$

• the **GHZ state** $|\psi\rangle = |0, \dots, 0\rangle + |1, \dots, 1\rangle$

$\underline{D=2:}$ $\quad \boldsymbol{A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0|\ ;\ A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1|}$

$$|\psi\rangle = \sum \mathrm{tr}[A^{i_1} A^{i_2} \dots A^{i_N}] |i_1, \dots, i_N\rangle \qquad \boldsymbol{\mathcal{P} = |0\rangle\langle 0,0| + |1\rangle\langle 1,1|}$$

• the **W state** $|\psi\rangle = |1,0,0,0,\dots\rangle + |0,1,0,0,\dots\rangle + |0,0,1,0,\dots\rangle + \dots$

$\underline{D=2:}$ $\quad \langle \boldsymbol{l}| = (0\ 1) = \langle \boldsymbol{1}|;\ \boldsymbol{A^0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1};\ \boldsymbol{A^1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |\boldsymbol{1}\rangle\langle \boldsymbol{0}|;\ |\boldsymbol{r}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\boldsymbol{0}\rangle$

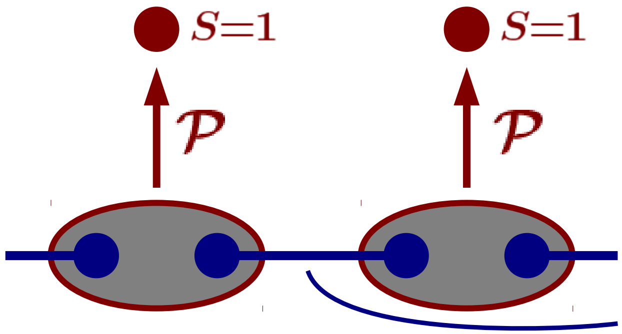
$$|\psi\rangle = \sum_{i_1, \dots, i_N} \langle l | A^{i_1} A^{i_2} \dots A^{i_N} | r \rangle |i_1, \dots, i_N\rangle$$

$\boldsymbol{\mathcal{P} = |0\rangle(\langle 0,0| + \langle 1,1|) + |1\rangle\langle 1,0|}$ (with $|1\rangle$ and $|0\rangle$ on left/right bnd.)

AKLT and RVB: rotationally invariant models

• The **AKLT state** [Affleck, Kennedy, Lieb & Tasaki, '87]

$\underline{D=2}$:

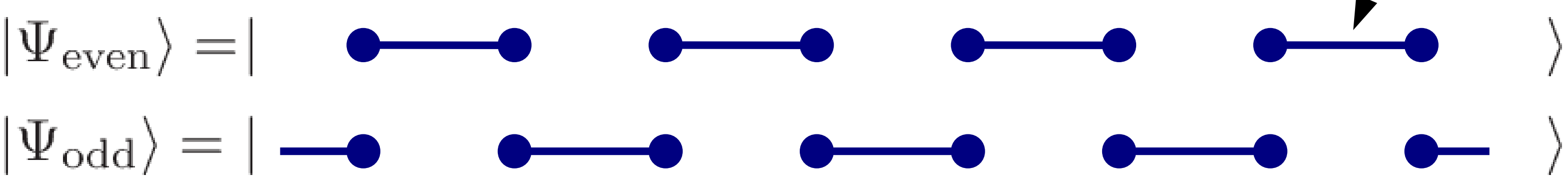


$|0,1\rangle - |1,0\rangle$ ← (different bond can be absorbed in $\mathcal{P}$)

$\mathcal{P}$: projector onto $S=1$ subspace
(symmetric subspace)

→ **rotationally invariant** model

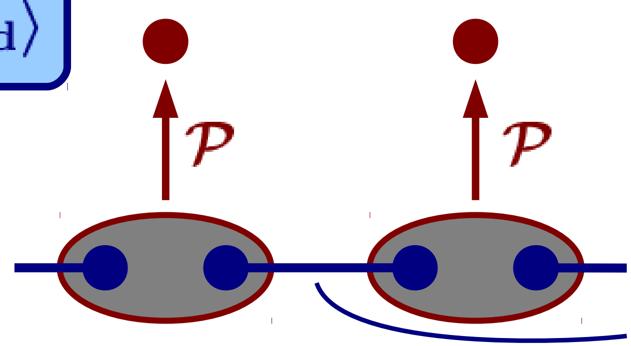
• **Resonating valence bond (RVB) state:** $|0,1\rangle - |1,0\rangle$



$$|\Psi_{\mathrm{RVB}}\rangle = |\Psi_{\mathrm{even}}\rangle + |\Psi_{\mathrm{odd}}\rangle$$

$$\mathcal{P} = |0\rangle\langle 0,2| + |1\rangle\langle 1,2| + |0\rangle\langle 2,0| + |1\rangle\langle 2,1|$$

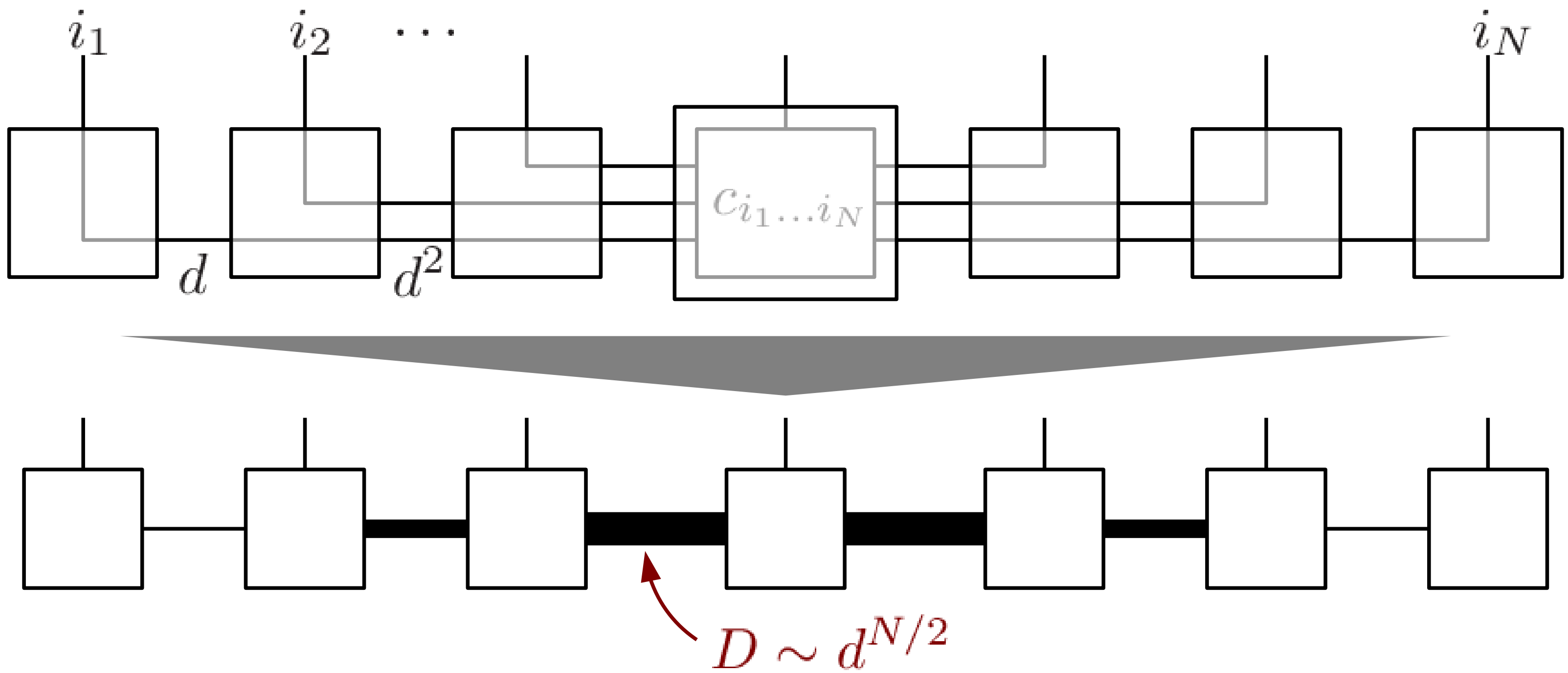
$\underline{D=3}$:



$|0,1\rangle - |1,0\rangle + |2,2\rangle$

When can we write state as MPS?

- Every state can be written as an MPS: $|\psi\rangle = \sum_{i_1,\dots,i_N} c_{i_1\dots i_N} |i_1,\dots,i_N\rangle$



- state with **entropic area law*** $S_\alpha(\rho_L) \leq S_{\max}$

  $\rightarrow$ efficient **MPS approximation** exists!

*: for **Renyi entropies**

$$S_\alpha(\rho) = \frac{\log \operatorname{tr} \rho^\alpha}{1-\alpha}, \quad \alpha < 1$$

$$\| |\Psi\rangle - |\mathrm{MPS}(D)\rangle \| \leq \mathrm{const} \times \frac{N \quad e^{c_\alpha S_{\max}}}{D^{c_\alpha}}$$

size N : linear scaling (poly if $S_{\max} \sim \log N$)

constant accuracy: $D \propto N^{1/c_\alpha}$