Quantum Information and Quantum Many-body Systems

– Lecture 3 –

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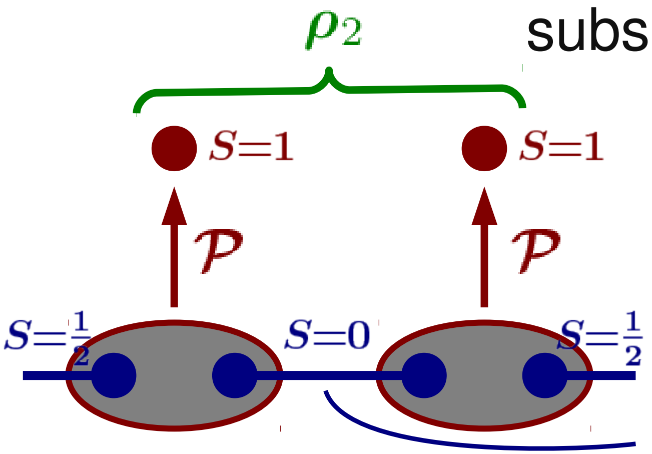
The AKLT Hamiltonian

• Let's consider again the **AKLT state:** [Affleck, Kennedy, Lieb, Tasaki]



$|1\rangle|0\rangle - |0\rangle|1\rangle$

subspace has spin $1 \otimes 1 = 0 \oplus 1 \oplus \cancel{2}$ **non-existent**

$\mathcal{P}$: projector onto $S=1$ subspace

• $\boldsymbol{h := \Pi_{S=2}}$: $h \geq 0$, and $h|\Psi_{\mathrm{AKLT}}\rangle = 0$

$\Rightarrow |\boldsymbol{\Psi_{\mathrm{AKLT}}}\rangle$ is a (frustration free) **ground state** of $\boldsymbol{H = \sum h_i}$

(frustration free = it minimized each h_i individually)

• It turns out: - $|\Psi_{\mathrm{AKLT}}\rangle$ is the **unique ground state** of H
- The Hamiltonian has a **spectral gap** above the ground state

• Note: we have that $h_i = \frac{1}{2}\left[\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + \frac{1}{3}(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2\right] + \frac{1}{3}$

More Hamiltonians for MPS

- The **GHZ state**

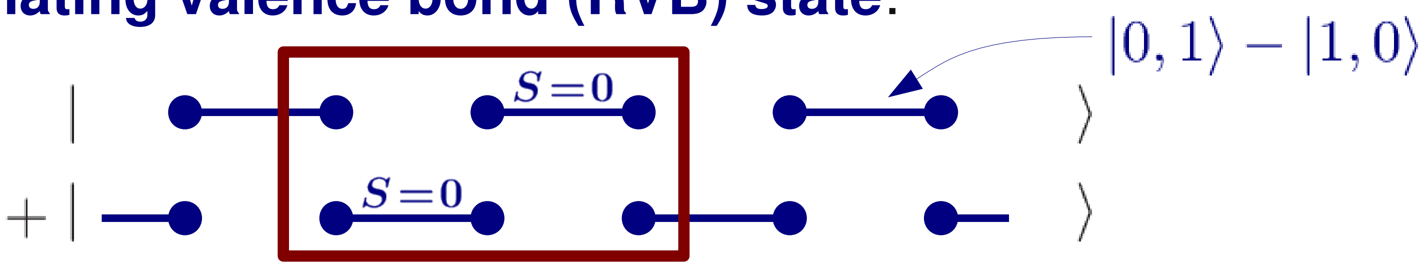
$$|\psi\rangle = |0,0,0,\ldots\rangle + |1,1,1\ldots\rangle \qquad \Rightarrow \qquad \rho_2 = |0,0\rangle\langle 0,0| + |1,1\rangle\langle 1,1|$$

$$\Rightarrow \boldsymbol{h_i = \Pi_{\ker \rho_2} = \tfrac{1}{2}(1 - Z_i Z_{i+1})}$$ **Ising Hamiltonian**

- Two-fold degenerate ground state, spectral gap

- The **resonating valence bond (RVB) state**:

$|\Psi_{\mathrm{RVB}}\rangle =$



$\boldsymbol{\rho_3}$: 3 spin- $\frac{1}{2}$ = $\frac{1}{2} \oplus \frac{1}{2} \oplus \cancel{\frac{3}{2}}$ **impossible**

$$\boldsymbol{h_i = \Pi_{S=3/2} = \tfrac{2}{3}(S_i \cdot S_{i+1} + S_i \cdot S_{i+2} + S_{i+1} \cdot S_{i+2}) + \tfrac{1}{2}}$$

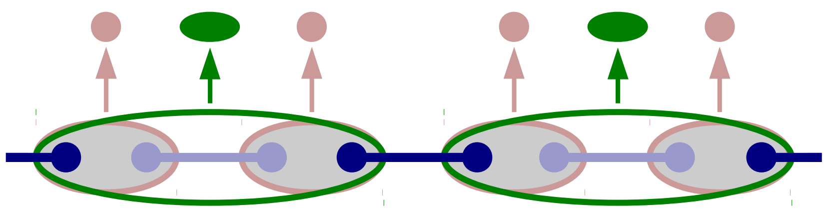
“Majumdar-Ghosh Hamiltonian”

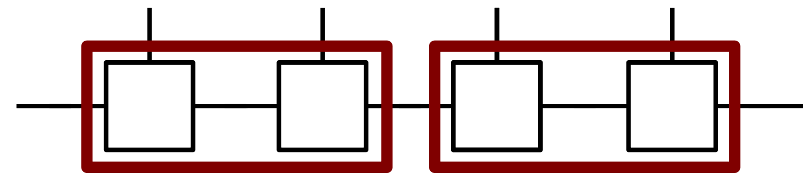
- at least two ground states (the two RVB configurations)

MPS and parent Hamiltonians

- Is there a Hamiltonian to any MPS $|\Psi\rangle$?

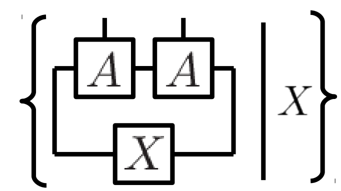
- MPS can be **blocked**:





- from now: consider **blocked MPS** s.th. $d \geq D^2$ (= $\log_d D^2$ sites/block)

- phys. state on 2 (blocked!) sites ρ_2
has rank at most $D^2 < d^2$: **rank deficit!**

$\mathrm{supp}(\rho_2) \subset \mathcal{S}_2 :=$ 

$= \{\mathrm{tr}[A^i A^j X] |i,j\rangle | X\}$

$\rightarrow$ **parent Hamiltonian** $H = \sum_i h_i$, with $\boxed{h = \mathbb{1} - \Pi_{\mathcal{S}_2}}$

- $H \geq 0\,,\ \ H|\Psi\rangle = 0$: $|\Psi\rangle$ is **ground state of H**

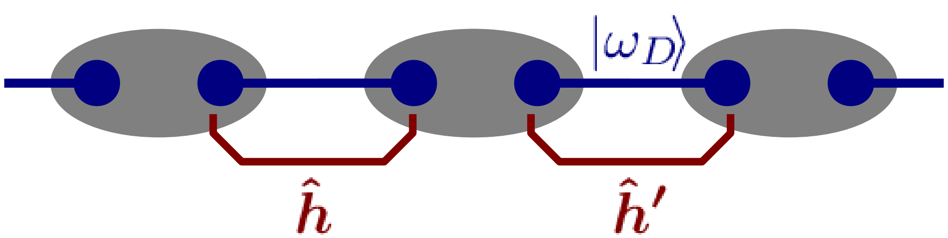
- What can we say about **ground state degeneracy?**

Uniqueness of the ground state

- Which MPS are **unique ground states** of their parent Hamiltonian?

- Consider "simplest possible MPS" $|\hat{\Psi}\rangle$:

$\hat{\mathcal{P}} = \mathbb{1}$ for some D



- parent Hamiltonian $\hat{H} = \sum_i \hat{h}_i$ has terms

$$\hat{h} = \mathbb{1} - |\omega_D\rangle\langle\omega_D| \qquad (\hat{h}|\hat{\Psi}\rangle = 0,\ \hat{h} \geq 0)$$

- Hamiltonian terms are commuting projectors: $[\hat{h}, \hat{h}'] = 0$

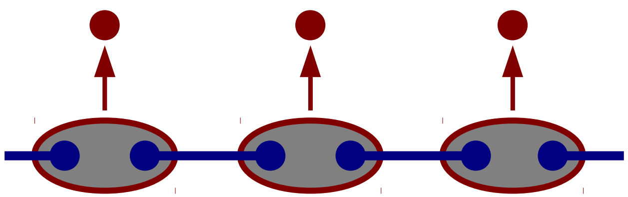
$\Rightarrow$ Hamiltonian has a **spectral gap**

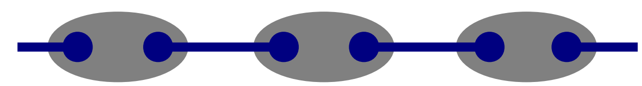
- one term $\hat{h}$ per bond $|\omega_D\rangle$ $\Rightarrow$ **unique ground state**!

Uniqueness of the ground state

- Does this generalize to MPS with more general $\mathcal{P}$?

- Consider MPS $|\Psi\rangle$ where $\mathcal{P}$ has inverse $\mathcal{P}^{-1}$ s.th. $\mathcal{P}^{-1}\mathcal{P} = \mathbb{1}$





$$|\Psi\rangle = \mathcal{P}^{\otimes N}|\omega_D\rangle^{\otimes D} \iff (\mathcal{P}^{-1})^{\otimes N}|\Psi\rangle = |\omega_D\rangle^{\otimes D} = |\hat{\Psi}\rangle$$

$$h := (\mathcal{P}^{-1} \otimes \mathcal{P}^{-1})\,\hat{h}\,(\mathcal{P}^{-1} \otimes \mathcal{P}^{-1})$$

$$h|\Psi\rangle = 0\,,\ h > 0 \iff \hat{h}|\hat{\Psi}\rangle = 0\,,\ \hat{h} > 0$$

$\Rightarrow$ $|\Psi\rangle$ **is the only ground state of** h **!**

- Fact: these Hamiltonians are **always gapped!**

- Note: h is not exactly the parent Hamiltonian h^0, but $c_1 h^0 \leq h \leq c_2 h^0$ holds
  $\Rightarrow$ uniqueness & gap are preserved

Injectivity (and beyond)

- A **(blocked) MPS** is called **injective** if:

$\mathcal{P}$ is injective $\Leftrightarrow$ $\mathcal{P}$ has a left-inverse $\mathcal{P}^{-1}\mathcal{P} = \mathbb{1}$ $\Leftrightarrow$ the A^i span the full matrix algebra

Injective MPS are unique ground state of their parent Hamiltonians!

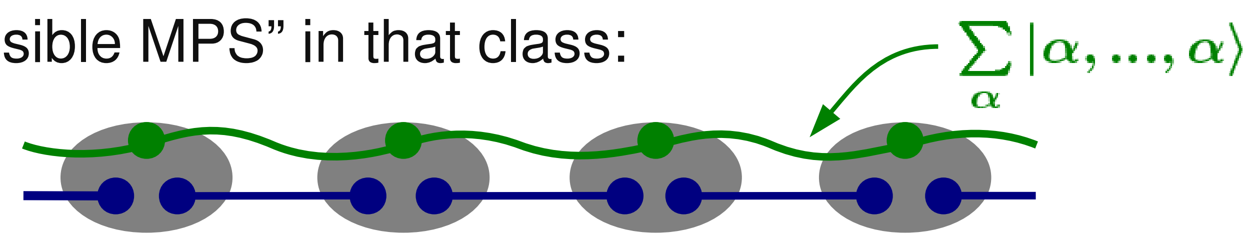
- Beyond injectivity:

The A_i span an algebra of **block-diagonal matrices** (with $\mathcal{A}$ blocks)

$\Rightarrow$ MPS is **$\mathcal{A}$-fold degenerate ground state** of its parent Hamiltonian!

(each block of the A_i describes one ground state)

- "simplest possible MPS" in that class:

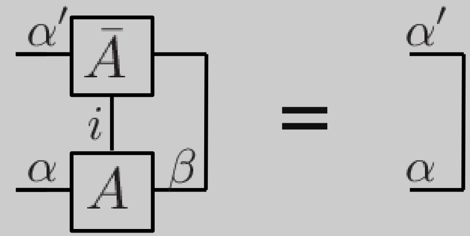


Any translational invariant MPS with periodic boundary conditions is of this form!

Standard form for MPS

- Any OBC MPS has **standard form** where $A^{[s],i}_{\alpha\beta} \equiv (A^{[s]})_{\alpha;i\beta}$ is an **isometry**

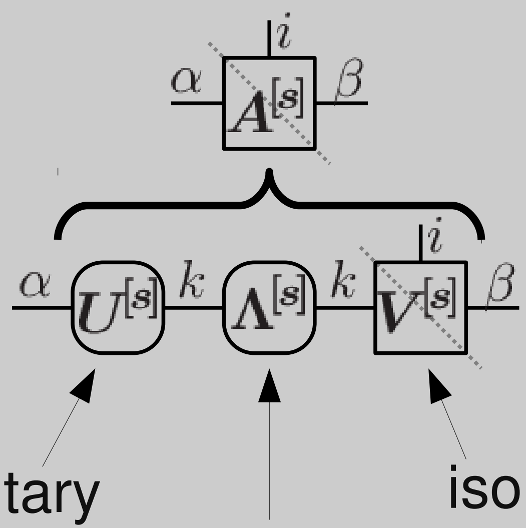
$$\sum_{i,\alpha} A^{[s],i}_{\alpha\beta}\, \bar{A}^{[s],i}_{\alpha'\beta} = \delta_{\alpha,\alpha'}$$

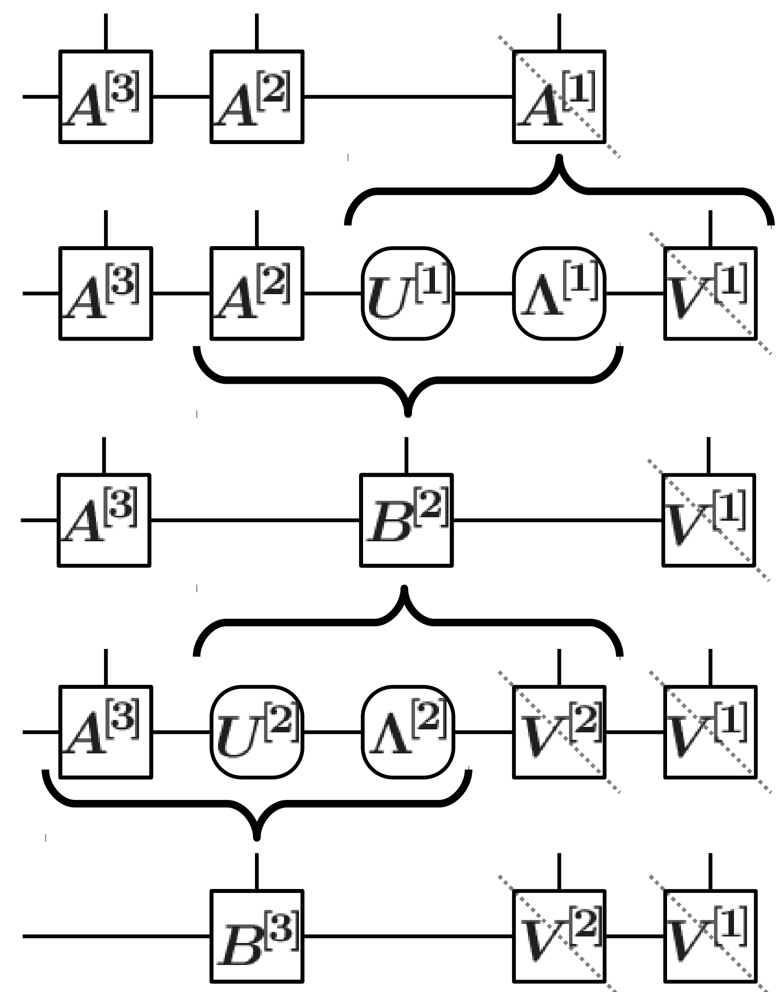


Proof:

Use *singular value decomposition:*

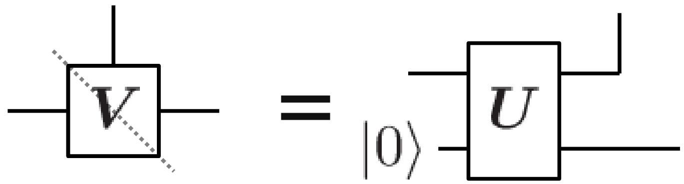
$$A^{[s],i}_{\alpha\beta} = \sum_k U^{[s]}_{\alpha,k} \Lambda^{[s]}_{kk} V^{[s]}_{k,(i\beta)}$$





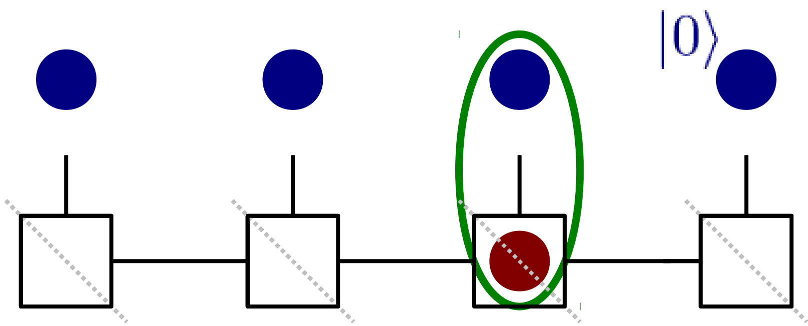
Sequential preparation of MPS

- Isometry = particle + ancilla + unitary evolution:



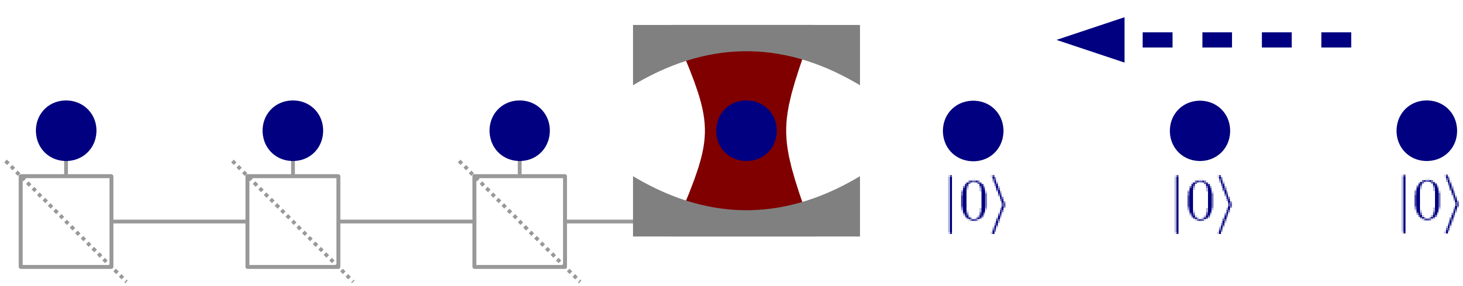
$$V|\phi\rangle = U|\phi\rangle|0\rangle$$

- MPS in normal form ↔ **sequential preparation** scheme



“hen and egg” scheme

- e.g.: beam of atoms interacting with a cavity



- various examples (GHZ, RVB, …) have a natural sequential interpretation!